1. FUNCTIONS AND MODELS

One of the important themes in calculus is the analysis of relationships between physical or mathematical quantities. Such relationships can be described in terms of graphs, formulas, numerical data, or words. In this chapter we will develop the concept of a "function," which is the basic idea that underlies almost all mathematical and physical relationships, regardless of the form in which they are expressed. We will study properties of some of the most basic functions that occur in calculus, including polynomials, exponential functions, and logarithmic functions.

1.1 Four Ways to Represent Functions

Many scientific laws and engineering principles describe how one quantity depends on another. This idea was formalized in 1673 by Gottfried Wilhelm Leibniz who coined the term function to indicate the dependence of one quantity on another.

**Example:** The area $A$ of a circle depends on the radius $r$ of the circle. The rule that connects $r$ and $A$ is given by the equation $A = \pi r^2$. With each positive number $r$ there is associated one value of $A$, and we say that $A$ is a function of $r$.

**Example:** Think about dropping a ball from a bridge. At each moment in time, the ball is a height above the ground. The height of the ball is a function of time. It was the job of physicists to come up with a formula for this function. This type of function is called real-valued since the "finished product" is a number (or, more specifically, a real number).
**Definition.** A *function* is a rule that assigns to each element $x$ in a set $D$ exactly one element, called $f(x)$, in a set $E$.

The set $D$ is called the *domain* of the function, which contains all values of the independent variable (typically $x$) that produce real values for the dependent variable (typically $y$).

The range of the function is the set of all possible $y$ values as $x$ varies throughout the domain. The number $f(x)$ is the *value of $f$ at $x$*, and is read "$f$ of $x$".

**Activity 1:** Describe the domain, range, independent variable and dependent variable in the above Examples.

**Activity 2:** Determine whether each of the following expressions represent a function?

(a) 

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

*Yes!*

(b) 

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

*No!*

(c) Amy is six years older than Betty. *No!*

(d) The rule which assigns to each car (at this exact point in time) the names of every person that has driven that car. *Yes!*
**Vertical Line Test:** An equation defines a function if each vertical line in the coordinate system passes through at most one point on the graph of the equation.

**Activity 3:** In each part of the accompanying figure, determine whether the graph defines $y$ as a function of $x$. 
Activity 4: Sketch a graph for each of the following equations and find the domain and range if it is a function.

(a) \( y - 2x = 1 \)

Yes!

\[ \text{Dom} = \mathbb{R} \]
\[ \text{Ran} = \mathbb{R} \]

(b) \( x^2 + y^2 = 1 \)

\[ y = \sqrt{1-x^2} \]
\[ y = -\sqrt{1-x^2} \]

No!

(c) \( y = x^2 + 3 \)

Yes!

\[ \text{Dom} = \mathbb{R} \]
\[ \text{Ran} = [3, \infty) \]
Activity 5: The graph of a function $f$ is shown below.

(a) What are the domain and range of $f$?

Domain = $[-1, 5]$  
Range = $[-2, 9]$  

(b) What are $f(-1), f(0), f(3), f(5)$?

\[ 0, \quad 4, \quad 7, \quad 9 \]
Activity 6: For $f(x) = 3x^2 - x + 2$, find the following:

(a) $f(2) = 3 \cdot (2)^2 - 2 + 2 = 3 \cdot 4 - 2 + 2 = 12$

(b) $f(-2) = 3 \cdot (-2)^2 - (-2) + 2 = 3 \cdot 4 + 2 + 2 = 16$

(c) $f(a+1) = 3 (a+1)^2 - (a+1) + 2 = 3 (a^2 + 2a + 1) - a - 1 + 2$
   $= 3a^2 + 6a + 3 - a + 1$
   $= 3a^2 + 5a + 4$

(d) $f(a+h) = 3 (a+h)^2 - (a+h) + 2 = 3 (a^2 + 2ah + h^2) - a - h + 2$
   $= 3a^2 + 6ah + 3h^2 - a - h + 2$

(e) $\frac{f(a+h) - f(a)}{h}$
   $\quad = \frac{3a^2 + 6ah + 3h^2 - a - h + 2 - (3a^2 - a + 2)}{h}$
   $\quad = \frac{6ah + 3h^2 - h}{h}$
   $\quad = \frac{h(6a + 3h - 1)}{h}$
   $\quad = 6a + 3h - 1$
Activity 7: Find the domain of each of the following functions.

(a) \( f(x) = \frac{1}{x-2} \)

\( f \) is undefined, so \( x \neq 2 \)

\( \text{Dom}(f) = (-\infty, 2) \cup (2, \infty) \)

(b) \( g(t) = \frac{5 + t}{\sqrt{t} - 1} \)

\( t - 1 > 0 \quad t > 1 \)

\( \text{Dom} = (1, \infty) \)

(c) \( f(x) = \frac{\sqrt{x-3}}{x-5} \)

\( x - 3 \geq 0 \quad \text{and} \quad x - 5 \neq 0 \)

\( x \geq 3 \quad \text{and} \quad x \neq 5 \)

\( \text{Dom} = [3, 5) \cup (5, \infty) \)

(d) \( f(x) = \frac{\sqrt{x-2}}{x^2 - 9} \)
**Definition.** Functions whose definition involve more than one rule are called \textbf{piecewise defined}.

To graph, graph each rule over the appropriate portion of the domain.

**Activity 8:** Graph $f(x) = |x|$.

\[
\begin{cases}
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases}
\]

**Activity 9:** Graph $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$
Definition.

• If a function $f$ satisfies $f(-x) = f(x)$ for every number $x$ in its domain, then $f$ is called an \textbf{even} function. The graph of an even function is symmetric with respect to \textit{y-axis}.

• If a function $f$ satisfies $f(-x) = -f(x)$ for every number $x$ in its domain, then $f$ is called an \textbf{odd} function. The graph of an odd function is symmetric with respect to \textit{origin}.

\textbf{Activity 10:} Determine whether each of the following functions is even, odd, or neither.

(a) $f(x) = \frac{x}{x^2 + 1}$

\[
f(-x) = \frac{-x}{(-x)^2 + 1} = -\frac{x}{x^2 + 1} = -f(x) \quad \rightarrow \text{Odd!}
\]

(b) $f(x) = 3x^2 - x^4$

\[
f(-x) = 3(-x)^2 - (-x)^4 = 3x^2 - x^4 = f(x) \quad \rightarrow \text{Even!}
\]

(c) $f(x) = \sqrt{x}$

\[
\begin{cases}
\sqrt{x} & x > 0 \\
\sqrt{-x} & x < 0
\end{cases}
\]

(d) $f(x) = x + |x|$

\[
f(-x) = -x + |-x| = -x + |x|
\]
Definition.

- A function $f$ is called **increasing** on an interval $I$ if $\ldots f(x_1) < f(x_2) \ldots$ whenever $x_1 < x_2$ on $I$.
- A function $f$ is called **decreasing** on an interval $I$ if $\ldots f(x_1) > f(x_2) \ldots$ whenever $x_1 < x_2$ on $I$.

**Activity 11:** For the function whose graph is given below, determine the interval(s) on which $f(x)$ is

(a) increasing $\left[-1, \frac{1}{4}\right), \left(3, 5\right]$ 

(b) decreasing $\left[\frac{1}{4}, 3\right]$