The Area Between Curves (cont.)

Recall that the area $A$ of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where $f$ and $g$ are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$, is given by

$$A = \int_a^b [f(x) - g(x)] \, dx$$

**Activity 3:** Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.

(a) Which car is ahead after one minute? Explain.

(b) What is the meaning of the area of the shaded region?

(c) Which car is ahead after two minutes? Explain.

(d) Estimate the time at which the cars are again side by side.
6.5 Average Value of a Function

Let \( s = s(t) \) denote the position function of a moving particle. In Section 2.1, we defined the average velocity \( v_{\text{ave}} \) of the particle over the time interval \([t_0, t_1]\) to be

\[
v_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}
\]

Let \( v(t) = s'(t) \) denote the velocity function of the particle. We saw in Section 5.2 that integrating \( s'(t) \) over a time interval gives the displacement of the particle over that interval, that is

\[
\int_{t_0}^{t_1} v(t) \, dt = \int_{t_0}^{t_1} s'(t) \, dt = s(t_1) - s(t_0)
\]

Thus,

\[
v_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} v(t) \, dt
\]

**Definition:** If \( f \) is continuous on \([a, b]\), then the average value (or mean value) of \( f \) on \([a, b]\) is defined to be

**Activity 1:** Find the average value of the function \( f(x) = x^2 + 1 \) on the interval \([-1, 2]\).
Activity 2: A glass of lemonade with a temperature of $40^\circ F$ is left to sit in a room whose temperature is a constant $70^\circ F$. Suppose that the temperature $T$ of the lemonade as a function of the elapsed time $t$ is modeled by the equation

$$T = 70 - 30e^{-0.5t}$$

where $T$ is in degrees Fahrenheit and $t$ is in hours. Find the average temperature $T_{ave}$ of the lemonade over the first 5 hours.
**Geometric Meaning of $f_{ave}$:** When $f$ is nonnegative on $[a, b]$, we have

$$f_{ave} \cdot (b - a) = \int_{a}^{b} f(x) \, dx$$

The left side of this equation is the area of a rectangle with a height of $f_{ave}$ and base of length $b - a$, and the right side is the area under $y = f(x)$ over $[a, b]$. Thus, $f_{ave}$ is the height of a rectangle constructed over the interval $[a, b]$, whose area is the same as the area under the graph of $f$ over that interval.

![Graph of f(x) and f_ave](image)

**Activity 3:** The velocity graph of an accelerating car is shown.

![Velocity graph](image)

(a) Estimate the average velocity of the car during the first 12 seconds.

(b) At what time was the instantaneous velocity equal to the average velocity?
The Mean Value Theorem for Integrals: If $f$ is continuous on $[a, b]$, then there exists a number $c$, $a \leq c \leq b$, such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

that is,

$$\int_a^b f(x) \, dx = f(c)(b-a)$$

Activity 3: Let $f(x) = (x - 3)^2$ be given.

(a) Find the average value of $f$ on $[2, 5]$.

(b) Find $c$ such that $f_{ave} = f(c)$.

(c) Sketch the graph of $f$ and a rectangle whose area is the same as the area under the graph of $f$ and the interval $[2, 5]$. 

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6.7 Applications to Economics and Biology

Consumer Surplus

Recall that the demand function \( p(x) \) is the price that a company has to charge in order to sell \( x \) units of a commodity. If \( X \) is the amount of the commodity that is currently available, then \( P = p(X) \) is the current selling price.

![Diagram of consumer surplus]

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price \( p \), corresponding to an amount demanded of \( x \), so it is given by the integral

\[
\text{Activity 1: } \text{The demand for a product, in dollars, is } p(x) = 1200 - 0.2x - 0.0001x^2. \text{ Find the consumer surplus when the sales level is } 500.
\]
**Blood Flow**

**Activity:** Suppose it is known that the velocity \( v \) of blood that flows along a blood vessel with radius \( R \) and length \( l \) at a distance \( r \) from the central axis is given by the formula

\[
v(r) = \frac{P}{4\eta l} (R^2 - r^2)
\]

where \( P \) is the pressure difference between the ends of the vessel and \( \eta \) is the viscosity of the blood.

(a) Find the formula for the rate of blood flow, or *flux* by evaluating the integral

\[
F = \int_{0}^{R} 2\pi rv(r)dr
\]

which is called *Poiseuille’s Law*.

(b) Use Poiseuille’s Law to calculate the rate of flow in a small human artery where we can take

\[
\eta = 0.027, \quad R = 0.008 \text{ cm}, \quad l = 2 \text{ cm}, \quad \text{and} \quad P = 4000 \text{ dynes/cm}^2.
\]