1.5 Exponential Functions

Definition. A function of the form $f(x) = b^x$, where $b > 0$, is called an exponential function with base $b$.

The function $f(x) = e^x$ is called the natural exponential function where $e \approx 2.718282$ (This notation was chosen by the Swiss mathematician Leonard Euler in 1727).

Activity 1: Graph the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$ in the same graph. Which function grows more quickly when $x$ is large?
Activity 2: Graph the functions $f(x) = 2^x$, $g(x) = 4^x$, $h(x) = \left(\frac{1}{2}\right)^x$, and $p(x) = \left(\frac{1}{4}\right)^x$, $q(x) = 1^x$ in the same graph, and explore the properties of the graphs of exponential functions:

\[
\left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x}
\]

Properties of the Graphs of $f(x) = a^x$

- Domain is the set of all real numbers.
- Range is the set of all positive real numbers.
- All graphs pass through the point $(0,1)$.
- Is any of the graphs not continuous (any holes or jumps)? No!
- The $x$-axis is a horizontal asymptote (but only in one direction).
- If $b$ is greater than 1, the graph is increasing (exponential growth).
- If $b$ is less than 1, the graph is decreasing (exponential decay).
Activity 3: Use the law of exponents to rewrite and simplify the expression.

(a) \( b^3 (2b)^4 = b^3 \cdot 2^4 \cdot b^4 = 2^4 \cdot b^{3+4} = 16 \cdot b^7 \)

(b) \( \frac{x}{\sqrt{x^3}} = \frac{x}{x^{3/2}} = x \cdot x^{-3/2} = x^{-1/2} = \frac{1}{\sqrt{x}} \)

Activity 4: Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?

I. One million dollars at the end of the month.

II. One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, \(2^{n-1}\) cents on the \(n\)-th day.
Activity 5: The population of a particular city doubles every 5 years. If 30,000 people currently live in the city,

(a) what will the population be in 15 years?

\[
\begin{align*}
P(0) &= 30,000 = P_o \\
P(5) &= 60,000 = P_o \cdot 2^1 \\
P(10) &= 120,000 = P_o \cdot 2^2 \\
P(15) &= 240,000 = P_o \cdot 2^3
\end{align*}
\]

(b) what will the population be in \(t\) years?

\[
P(t) = P_o \cdot 2^{t/5}
\]

(c) estimate the size of the population in 27 years.

\[
P(27) = P_o \cdot 2^{27/5} = 30,000 \times 2^{27/5}
\]

\[
= 1,266,727.594
\]
1.6 Inverse Functions and Logarithms

**Definition.** A function $f$ is said to be **one-to-one** if each range value corresponds to exactly one domain value.

**Horizontal Line Test:** If every horizontal line intersects the graph of a function $f$ in no more than one place, then $f$ is a one-to-one function.

**Activity 1:** Which one(s) of the below functions are one-to-one?
**Definition.** If \( f \) is a one-to-one function with domain \( A \) and range \( B \), then its **inverse function** \( f^{-1} \) has domain \( B \) and range \( A \) and is defined by

\[
f^{-1}(y) = x \iff f(x) = y
\]

for any \( y \) in \( B \). Thus, if \((a, b)\) is a point on the graph of \( f \), then \((b, a)\) is a point on the graph of \( f^{-1} \).

Thus, if \( f \) is one-to-one, then the graph of its inverse, \( f^{-1} \), is a **reflection about the line** \( y = x \).

**Activity 2:** If \( f \) and \( g \) are one-to-one functions such that \( f(3) = 5 \) and \( g^{-1}(2) = 3 \), what is \( g \circ f^{-1}(5) \)?

**How to Find the Inverse Function of a One-to-One Function \( f \):**

**Step 1.** Write \( y = f(x) \).

**Step 2.** Solve this equation for \( x \) in terms of \( y \) (if possible).

**Step 3.** To express \( f^{-1} \) as a function of \( x \), interchange \( x \) and \( y \). The resulting equation is \( y = f^{-1}(x) \).

**Activity 3:** Find the inverse of \( f(x) = 1 + \sqrt{2-x} \). What are the domain and range of \( f \) and \( f^{-1} \)? Graph both functions and observe that \( f^{-1} \), is a reflection about the line \( y = x \).
**Definition.**  The inverse of an exponential function \((y = b^x)\) is called a **logarithmic function**. For \(b > 0\) and \(b \neq 1\),

\[ y = \log_b x \iff b^y = x. \]

**Activity 4:** Solve for the following equations for \(x\), \(y\), or \(b\) without a calculator.

(a) \(\log_4 x = 2 \quad \Rightarrow \quad x = 4^2\)

(b) \(\log_b e^{-3} = -3 \quad \Rightarrow \quad (b^{-\frac{3}{2}}) = (e^{-3}) \quad \Rightarrow \quad b = e\)

(c) \(\log_{25}(\frac{1}{5}) = y \quad \Rightarrow \quad 25^y = \frac{1}{5} \quad \Rightarrow \quad (5^2)^y = 5^{-1} \quad \Rightarrow \quad 5^{2y} = 5^{-1} \quad \Rightarrow \quad 2y = -1 \quad \Rightarrow \quad y = -\frac{1}{2}\)

**Properties of Logarithmic Functions**

- \(\log_b 1 = y \quad \Rightarrow \quad b^y = 1 \quad \Rightarrow \quad y = 0 \quad \therefore \quad \log_b 1 = 0\)
- \(\log_b b = 1\)
- \(\log_b b^x = y \quad \Rightarrow \quad b^y = b^x \quad \Rightarrow \quad y = x \quad \therefore \quad \log_b b^x = x\)
- \(b^{\log_b x} = z \quad \Rightarrow \quad \log_b b^x = \log_b z \quad \Rightarrow \quad x = z \quad \therefore \quad b^{\log_b x} = x\)
Activity 5: Use the laws of logarithm to simplify the following expressions.

(a) $\log_2 6 - \log_2 15 + \log_2 20 = \log_2 \left( \frac{6 \cdot 20}{15} \right)$

(b) $\log_5 (3 + x^2) - 2 \log_5 x - \log_5 25 = \log_5 \left( \frac{3 + x^2}{x^2 \cdot 25} \right)$

Common Logarithm: Logarithms with base 10 are called common logarithms and are often written without explicit reference to the base. Thus, the symbol $\log x$ generally denotes $\log_{10} x$.

Natural Logarithm: $\log_e x = \ln x$

- $\ln x = y \Leftrightarrow e^y = x$
- $\ln (e) = x, x \in \mathbb{R}$
- $e^{\ln x} = x, x > 0$
- $\ln e = 1$

Change of base Formula For any positive number $a$ ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}$$

Activity 6: Find the exact value of $(\log_2 81)(\log_3 32)$ without using a calculating utility.

$$81 = 3^4 \quad \Rightarrow \quad (\log_2 81)(\log_3 32) = \log_2 3^4 \cdot \log_3 2^5$$

$$= 4 \left( \frac{\ln 3}{\ln 2} \cdot \frac{\ln 2}{\ln 3} \right)$$

$$= 2 \cdot \frac{\ln 3}{\ln 2} \cdot \frac{\ln 2}{\ln 3} = 20$$
Activity 7: Evaluate the following using a calculator.

(a) \( \log 0.74 = \color{red}{-0.1304} \)

(b) \( \ln \pi = \color{red}{1.144} \)

Activity 8: Solve each equation for \( x \).

(a) \( 8y = 3^x \)

\[
\log_3 8y = \log_3 3^x = x
\]

(b) \( e^y = 5^{-x} \)

\[
\log_5 e^y = \log_5 5^{-x} = -x
\]

\[
x = - \log_5 e^y = -y \log_5 e = -y \frac{\ln e}{\ln 5} = -\frac{y}{\ln 5}
\]

(c) \( \ln 4x - 3 \ln(x^2) = \ln 2 \)

\[
\ln 4x - \ln (x^2)^3 - \ln 2 = 0
\]

\[
\ln \left( \frac{4x}{x^6 \cdot 2} \right) = 0
\]

\[
\frac{4x}{2 \cdot x^6} = 1 \implies x^5 = 2 \implies x = \sqrt[5]{2}
\]

Activity 9: 25 rabbits are introduced to an island, where they quickly reproduce and the rabbit population grows according to an exponential model \( P(t) = P_0 e^{kt} \) so that the population doubles every four months. If \( t \) is in months, what is the value of the continuous growth rate \( k \)?

\[
P(0) = 25 = P_0
\]

\[
P(4) = P_0 \cdot 2 = 50
\]

\[
P(4) = 25 e^{4k} = 50 \implies \ln e^{4k} = \ln 2
\]

\[
4k = \ln 2 \implies k = \frac{\ln 2}{4}
\]