2.6 Derivatives and Rates of Change

**Preview Activity:** Suppose that $f$ is the function given by the graph below and that $a$ and $a + h$ are the input values as labeled on the $x$-axis. Use the below graph to answer the following questions.

(a) Locate and label the points $(a, f(a))$ and $(a + h, f(a + h))$ on the graph.

(b) Construct a right triangle whose hypotenuse is the line segment from $(a, f(a))$ to $(a + h, f(a + h))$. What are the lengths of the respective legs of this triangle?

(c) What is the slope of the line that connects the points $(a, f(a))$ and $(a + h, f(a + h))$?

$$
\tan \theta = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a + h) - f(a)}{h}
$$

(d) Write a meaningful sentence that explains how the average rate of change of the function on a given interval and the slope of a related line are connected.

“**The average rate of change of $f$ on an interval $[a,b]$ is the slope of the line that passes through the points $(a, f(a))$ and $(b, f(b))$.”**

(e) What happens to the line that connects the points $(a, f(a))$ and $(a + h, f(a + h))$ as $h$ gets smaller and smaller?

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touches the graph of $f$ at $a$.
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(f) Conjecture a formula for the slope of the line that touches the graph of $f$ at the point $(a, f(a))$?

$$
m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
$$
**Definition:** The **tangent line** to the curve \( y = f(x) \) at the point \( P(a, f(a)) \) is the line through \( P \) with slope

\[
m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

provided that this limit exists.

**Activity 1:** Find an equation of the tangent line to the parabola \( y = x^2 \) at the point \( P(1, 1) \).

\[
m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} \]

\[
= \lim_{x \to a} \left( \frac{(x+a)(x-a)}{x-a} \right) = 2a
\]

\[m \neq P(1,1) = 2\]

\[y = 2x + n \Rightarrow n = y - 2x = 1 - 2 = -1 \quad y = 2x - 1\]

**Activity 2:** Recall that in Section 2.1, we tried to compute the instantaneous velocity of a falling ball whose position function given as \( f(t) = 64 - 16(t - 1)^2 \).

(a) Write a formula that gives the instantaneous velocity of the ball at time \( t = a \) in terms of the tangent lines.

**instantaneous velocity at \( t=a \) is the slope of the tangent line to the graph of \( f \) at the point \( (a, f(a)) \). So,**

\[
m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

(b) Use your formula to compute the instantaneous velocity at time \( t = 2 \). Compare your results with the previous ones.

\[
m = \lim_{t \to 2} \frac{f(t) - f(2)}{t - 2} = \lim_{t \to 2} \frac{64 - 16(t^2 - 2t + 1) - 49}{t - 2}
\]

\[
= \lim_{t \to 2} \frac{1 - (t-1)^2}{t - 2} = \lim_{t \to 2} \frac{[(t^2 - 2t + 1) + 1]}{t - 2}
\]

\[
= \lim_{t \to 2} \frac{1}{t - 2}
\]

(c) How fast is the ball traveling when it hits the ground?

\[
m = \lim_{t \to 3} \frac{f(t) - f(3)}{t - 3} = \lim_{t \to 3} \frac{64 - 16(t^2 - 2t + 1)}{t - 3}
\]

\[
= \lim_{t \to 3} \frac{16(t - 3)(t - 1)}{t - 3} = \lim_{t \to 3} 16 \cdot \frac{4 - (t-1)^2}{t - 3}
\]

\[
= 16 \cdot \lim_{t \to 3} \frac{2 -(t-1)(2t+1)}{t - 3} = (16 \lim_{t \to 3} \frac{2}{1} - 1) \cdot \frac{1}{3} = 64
\]