2.7 The Derivative as a Function

Recall that the instantaneous rate of change of a function $f$ at a point $a$ is the slope of the tangent line to the graph of $f$ at the point $(a, f(a))$, which is given by the limit

$$
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
$$

Since this type of a limit arises whenever we calculate a rate of change in any of the sciences or engineering, and it occurs so widely, it is given a special name and notation.

**Definition:** The derivative of a function $f$ at a point $a$ is given by the formula

$$
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
$$

If we let $a$ vary and replace it by a variable $x$ then we obtain a new function

$$
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
$$

which is called the derivative of $f$.

Other Notations for the derivative of $y = f(x)$:

$$
f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)
$$

**Activity 1:** If $f(x) = \sqrt{x}$, find the derivative of $f$. State the domain of $f'$. 

\[\text{Activity 1} \quad \text{If } f(x) = \sqrt{x}, \text{ find the derivative of } f. \text{ State the domain of } f'. \]
**Definition:** A function $f$ is **differentiable at a point** $a$ if ............ It is differentiable on an open interval $(a, b)$ if it is differentiable at ......................

Geometrically, a function is differentiable at a point $a$, if the graph of the function passes “smoothly” through that point.

**Activity 2:** Try to draw tangent lines at the point $x = a$ for each graph given below.

If $f'(a)$ does not exist, then we say that $f(x)$ is **non-differentiable** at $x = a$. This occurs when the graph has:

1. 
2. 
3. 

**Activity 3:** Explore the relationship with differentiability and continuity, and fill in the blanks of the below sentence using the words *continuous* and *differentiable*, when appropriate.

If $f$ is ..................................... at a point $a$ then it must be ........................................ at $a$.

**Activity 4:** The graph of a function $f(x)$ is given. Sketch a rough graph of $y = f'(x)$.
Activity 5: Match the graphs of the functions shown in (a)-(f) with the graphs of their derivatives in (A)-(F).
**Definition:** If $f$ is a differentiable function, then its derivative $f'$ is also a function, so $f'$ may have a derivative of its own, denoted by $(f')' = f''$. This new function $f''$ is called the **second derivative** of $f$ because it is the derivative of the derivative of $f$.

\[ f''(x) = y'' = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} \]

**Position, Velocity, and Acceleration:**

- The position function of an object that moves in a straight line is usually given by $s(t)$.

- The velocity of an object as a function of time is given by:

- The acceleration of an object is the instantaneous rate of change of velocity with respect to time and is given by:

**Activity 6:** The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.

![Graph of three functions](image)
2.8 What Does $f'$ Say about $f$?

**Preview Activity:** In each part, use the graph of $y = f(x)$ in the accompanying figure to find the requested information.

![Graph of $y = f(x)$ with points at x=1, 2, 3, 4, 5, 6, 7]

(a) Find the intervals on which $f$ is increasing.

(b) Find the intervals on which $f$ is decreasing.

(c) Find the intervals on which $f'$ is positive.

(d) Find the intervals on which $f'$ is positive.

**Activity 1:** Fill in the blanks in the following sentences.

If $f'(x) > 0$ on an interval, then $f$ is ................. on that interval.

If $f'(x) < 0$ on an interval, then $f$ is ................. on that interval.

If $f'(x) = 0$ on an interval, then $f$ is ................. on that interval.
**Activity 2:** The graph of $f'$ is given. On what intervals is $f$ decreasing, increasing?

**Definition:**

- We say that $f$ has a **local maximum** at a point $a$, if the derivative of $f$ changes sign from positive to negative at $a$.

- We say that $f$ has a **local minimum** at a point $a$, if the derivative of $f$ changes sign from negative to positive at $a$.

**Activity 3:** Find local maxima and local minima of the function $f$ whose derivative is given in Activity 2.
**Definition:**

- The graph of a function $f$ is **concave upward** on the interval $(a, b)$ if .........................

- The graph of a function $f$ is **concave downward** on the interval $(a, b)$ if .........................

- A point $a$ is called an **inflection point** of $f$ if ................. Graphically, its where ................. ........................................

**Activity 4:** The graph of $f$ is given below. Find the intervals where $f$ is increasing/decreasing, concave up/concave down, find all local maximum/minimum points, find all inflection points.
Activity 5: Sketch the graph of a function which satisfies the following

- \( f'(x) > 0 \) for all \( x \neq 1 \)
- has a vertical asymptote at \( x = 1 \)
- \( f''(x) > 0 \) if \( x < 1 \) or \( x > 3 \)
- \( f''(x) < 0 \) if \( 1 < x < 3 \)