Let V be the linear space consisting of all polinomials of degree at most 2 on the interval $[0,1]$ of the real line. Equip V with the following inner product

$$\langle p, q\rangle = \int_0^1 x^2 p(x) q(x) dx\,.$$

Orthonormalize the canonical basis $1, x, x^2$ with respect to the previous inner product.

1. SOLUTION

Put $\mathbf{v}_j = x^j,\ j = 0, 1, 2$. Denote $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\}$ our orthonormal set. Then

$$\mathbf{u}_0 = \frac{\mathbf{u}_0}{\|\mathbf{u}_0\|}\,.$$

$$\mathbf{w}_1 = \mathbf{v}_1 - \langle \mathbf{v}_1, \mathbf{u}_0\rangle \mathbf{u}_0\,,\ \ \mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}\,.$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\langle \mathbf{v}_2, \mathbf{u}_0\rangle \mathbf{u}_0 + \langle \mathbf{v}_2, \mathbf{u}_1\rangle \mathbf{u}_1\right)\,,\ \ \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}\,.$$

Thus

$$\mathbf{u}_0 = \frac{1}{\sqrt{\int_0^1 x^2 dx}} = \sqrt{3}\,.$$

$$\mathbf{w}_1 = x - 3\int_0^1 x^3 dx = x - \frac{3}{4}\,,$$

$$\mathbf{u}_1 = \frac{x - \frac{3}{4}}{\sqrt{\int_0^1 x^2 (x - \frac{3}{4})^2 dx}} = \sqrt{5}(4x-3)\,.$$

$$\mathbf{w}_2 = x^2 - 5(4x-3)\int_0^1 x^4(4x-3)dx - 3\int_0^1 x^4 dx = x^2 - \frac{4}{3}x + \frac{2}{5}\,,$$

$$\mathbf{u}_2 = \frac{x^2 - \frac{4}{3}x + \frac{2}{5}}{\sqrt{\int_0^1 x^2 (x^2 - \frac{4}{3}x + \frac{2}{5})^2 dx}} = 15\sqrt{\frac{2}{11}}(x^2 - \frac{4}{3}x + \frac{2}{5})\,.$$