COMPLEMENTS

the use of the step–function

Consider for $t > 0$ the following function

$$f(t) = \begin{cases} f_0(t), & 0 < t < t_1 \\ f_1(t), & t_1 < t < t_2 \\ . \\ . \\ f_j(t), & t_j < t < t_{j+1} \\ . \\ . \end{cases}$$

Let $f_j : (0, +\infty) \mapsto \mathbb{R}$, $j = 1, 2, \dots$ be functions defined on the half–line $(0, +\infty)$.

Let u be the one–step function given by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 < t \end{cases} \tag{0.1}$$

Then

$$f(t) = f_0(t) + \sum_j u(t - t_j)[f_j(t) - f_{j-1}(t)] \, .$$

The proof of the above formula is just a simple computation by looking at successive intervals (t_{j-1}, t_j).

the Laplace transform

The Laplace transform of the function $f : [0, +\infty) \mapsto \mathbb{R}$ is defined as

$$\mathcal{L}\{f\}(s) := \int_0^{+\infty} e^{-st} f(t) dt \, ,$$

for the $s \in \mathbb{R}$ such that the above integral is meaningful. We denote the Laplace transform of the function $f, g, h, \dots$, as $F(s), G(s), H(s), \dots$.

We have

$$\mathcal{L}\{af+bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}\,,$$
$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)\,,$$
$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\,,$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}(s)\,.$$

Here there is a list of the Laplace transform of elementary functions

$$\mathcal{L}\{1\}(s) = \frac{1}{s}\,, \qquad s>0\,,$$
$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}\,, \qquad s>a\,,$$
$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}\,, \qquad s>0\,,\ \ n=0,1\ldots\,,$$
$$\mathcal{L}\{e^{at}t^n\}(s) = \frac{n!}{(s-a)^{n+1}}\,, \qquad s>a\,,\ \ n=0,1\ldots\,,$$
$$\mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2+b^2}\,, \qquad s>0\,,$$
$$\mathcal{L}\{\cos bt\}(s) = \frac{s}{s^2+b^2}\,, \qquad s>0\,,$$
$$\mathcal{L}\{e^{at}\sin bt\}(s) = \frac{b}{(s-a)^2+b^2}\,, \qquad s>a\,,$$
$$\mathcal{L}\{e^{at}\cos bt\}(s) = \frac{s-a}{(s-a)^2+b^2}\,, \qquad s>a\,.$$

The Laplace transform of other elementary functions can be obtained by using the rules listed above.

The inverse Laplace transform $\mathcal{L}^{-1}$ is linear

$$\mathcal{L}^{-1}\{af+bg\} = a\mathcal{L}^{-1}\{f\} + b\mathcal{L}^{-1}\{g\}\,.$$

We can use the properties listed below (and some other stuff in the textbook such as the **method of partial fractions**), to compute the inverse Laplace transform in many useful situations.

If $s>a$, useful rules are

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\}(t) = e^{at}t^n\,,$$

$$\mathcal{L}^{-1}\left\{\frac{d^nF}{ds^n}\right\}(t) = (-t)^n f(t)\,.$$

Let u be the one–step function given (0.1). Then

$$\begin{aligned}
&\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)\,,\\
&\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)\,,\\
&\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)\,.
\end{aligned}$$

A function f is periodic of period T if

$$f(t) = f(t+T)$$

for all t in the domain of f.

If $f : [0,+\infty) \mapsto \mathbb{R}$ is piecewise continuous and periodic of period T, then

$$\mathcal{L}\{f\}(s) = \frac{\int_0^T e^{-st}f(t)dt}{1-e^{-Ts}}\,.$$

The convolution of f and g is defined as

$$(f*g)(t) := \int_0^t f(t-v)g(v)dv\,,$$

provided the integral in the r.h.s. is meaningful. One has

$$\begin{aligned}
&\mathcal{L}\{(f*g)\}(s) = F(s)G(s)\,,\\
&\mathcal{L}^{-1}\{FG\}(t) = (f*g)(t)\,.
\end{aligned}$$

Let δ be the distribution acting on the space $C([0,+\infty))$ made of all the continuos functions on the positive half–line $[0,+\infty)$. It is defined as

$$\int_0^{+\infty} f(t)\delta(t-a)dt = f(a)\,, \quad a \geq 0\,.$$

Then

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}\,, \quad a \geq 0\,.$$

the Cramer rule

We treat the 2 or 3 dimensional case, the general case follows analogously.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ ($A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$) be a 2×2 (3×3) matrix with $\det A \neq 0$, and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ ($\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$) a fixed 2–dimensional (3–dimensional) column vector.

The Cramer formula for the inverse matrix A^{-1} is given by.

$$(A^{-1}\mathbf{b})_1 = \frac{\det \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det A}, \qquad (A^{-1}\mathbf{b})_2 = \frac{\det \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det A}.$$

$$(A^{-1}\mathbf{b})_1 = \frac{\det \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}}{\det A}, \qquad (A^{-1}\mathbf{b})_2 = \frac{\det \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}}{\det A},$$

$$(A^{-1}\mathbf{b})_3 = \frac{\det \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}}{\det A}.$$

differential linear systems

Let

$$\mathbf{x}'(t) = A\mathbf{x}(t) \tag{0.2}$$

be a differential linear homogeneous system with constant coefficients, where A is a $n \times n$ numerical matrix. λ is a *eigenvalue* of A if it solves

$$\det(\lambda I - A) = 0\,. \tag{0.3}$$

An *eigenvector* $\mathbf{u}_\lambda$ with eigenvalue λ is a nontrivial solution of the equation

$$A\mathbf{u}_\lambda = \lambda \mathbf{u}_\lambda$$

which always exists thanks to (0.3). Suppose that A admits a system of n linearly independent eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ corresponding to real eigenvalues $\lambda_1, \dots, \lambda_n$, some of them appearing possibly, with multiplicity. Then a set of linearly independent solutions for (0.2) are

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1, \dots, \mathbf{x}_n(t) = e^{\lambda_n t}\mathbf{u}_n\,.$$

The general solution of (0.2) is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{u}_n \tag{0.4}$$

and the *fundamental matrix* $X(t)$ of solutions is defined as

$$X(t) := \mathrm{col}(e^{\lambda_1 t}\mathbf{u}_1 \cdots e^{\lambda_n t}\mathbf{u}_n)\,. \tag{0.5}$$

It is an everywhere invertible matrix. If A in (0.2) admits a complex eigenvalue $\lambda = \alpha + i\beta$ with complex eigenvector $\mathbf{u} = \mathbf{a} + i\mathbf{b}$, then $\overline{\lambda} = \alpha - i\beta$ is an eigenvalue too, as A has real entries. The corresponding eigenvector is $\overline{\mathbf{u}} = \mathbf{a} - i\mathbf{b}$. A pair on linearly independent real solutions of (0.2) are

$$e^{\alpha t}[\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}]\,, \qquad e^{\alpha t}[\sin(\beta t)\mathbf{a} + \cos(\beta t)\mathbf{b}]\,. \tag{0.6}$$

By replacing the linearly complex conjugate solutions $e^{\lambda t}\mathbf{u}$, $e^{\overline{\lambda} t}\overline{\mathbf{u}}$ in (0.4) with the corresponding real linearly independent ones, (0.4) and (0.5) are still meaningful.

Let

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{0.7}$$

be a differential linear (non homogeneous) system in a n–dimensional space. Suppose that $(\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t))$ are n linearly independent solutions of

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)\,. \tag{0.8}$$

The fundamental matrix $X(t)$ of solutions of the homogeneous associated system (0.8) is defined as in (0.5) by

$$X(t) := \mathrm{col}(\mathbf{x}_1(t) \cdots \mathbf{x}_n(t))\,.$$

Then the general solution of (0.7) is

$$\mathbf{x}(t) = X(t)\left[\mathbf{c} + \int X^{-1}(t)\mathbf{f}(t)dt\right],$$

where $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{pmatrix}$ is a column vector of free constants.

If we have the following initial value problem

$$\begin{cases} \mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)\,, \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{0.9}$$

then the solution of (0.9) is given by

$$\mathbf{x}(t) = X(t)\left[X^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} X^{-1}(s)\mathbf{f}(s)ds\right].$$