Embedding of ℓ_1 into Lipschitz-free Banach spaces and ℓ_∞ into their duals

Marek Cúth

Metric Spaces: Analysis, Embeddings into Banach Spaces, Applications

References

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Definition

Given $(M, \rho, 0)$, we define the *Lipschitz-free space over M* by

$$\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta(m) :\ m \in M\} \subset \mathrm{Lip}_0(M)^*.$$

Universal property

Proposition (Universal property)

Let $(M, \rho, 0)$ be as above, X a Banach space and $L : M \to X$ a Lipschitz mapping with $L(0) = 0$. Then there is a unique linear operator $\hat{L} : \mathcal{F}(M) \to X$ with $\hat{L}\delta = L$ and $\|\hat{L}\| = \|L\|_{\mathrm{Lip}}$.

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- In the following picture, all the arrows commute:

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Embedding of ℓ_∞ into $\mathrm{Lip}_0(M)$

First idea: Consider functions $f_n(x) = \max\{r_n - \rho(x, x_n), 0\}$ with disjoint supports; i.e., $U(x_n, r_n)$ are pairwise disjoint.

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Relation with the embedding of ℓ_1 into $\mathcal{F}(M)$: [Bessaga, Pełczyński]
$\ell_\infty \hookrightarrow X^*$ if and only if $\ell_1 \overset{c}{\hookrightarrow} X$.

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Theorem

Let M be an infinite metric space. For the Banach space $X = \mathcal{F}(M)$, we have

(i) $\ell_1 \overset{c}{\hookrightarrow} X$, *i.e., there is a complemented subspace of X isomorphic to ℓ_1.*

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From this we get

(ii) $X \overset{c}{\not\hookrightarrow} \mathcal{C}(K)$, *i.e., X is not isomorphic to a complemented subspace of a $\mathcal{C}(K)$ space.*

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(iv) X *is not isomorphic to the Gurariĭ space.*

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(iv) *X is not isomorphic to the Gurariĭ space.*

(v) *X is a projectively universal separable Banach space, i.e., for any separable Banach space Y there exists a bounded linear operator from X onto Y.*

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Let X be a Banach space. Denote by $\mathcal{P}_F(X)$ the set of equivalent norms $|\cdot|$ on X for which there is a metric space M with $\mathcal{F}(M)$ isometric with $(X, |\cdot|)$. Then $\mathcal{P}_F(X)$ is of first category in $\mathcal{P}(X)$ (i.e. in the space of all equivalent norms on X).

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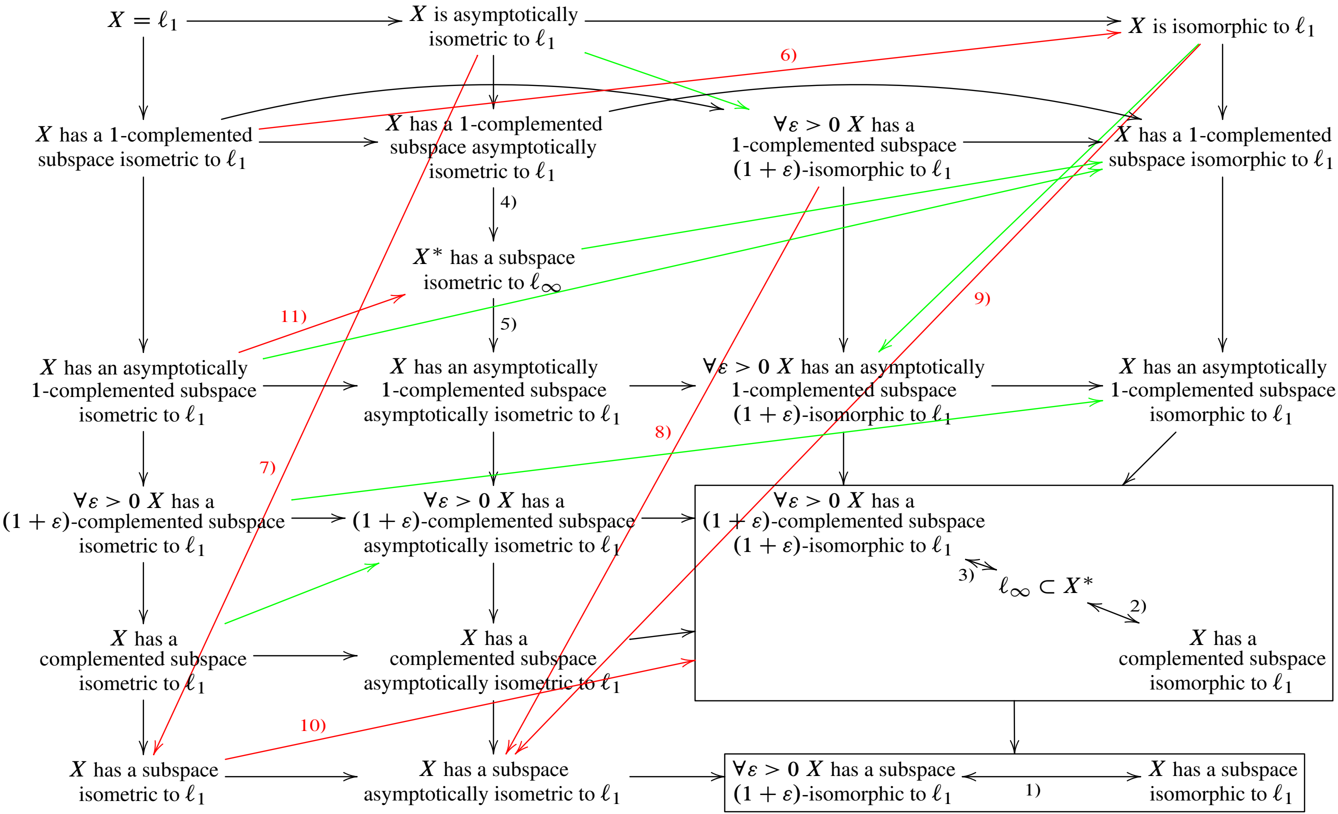
(on the other hand, there are metric spaces $M = \{0, x, y\}$ such that ℓ_1^2 does not embed linearly isometrically into $\mathcal{F}(M)$).

Embeddings of ℓ_1 into a general Banach space X

The end

Thank you for your attention!



$X = \ell_1$
X is asymptotically isometric to ℓ_1
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4)
X^* has a subspace isometric to ℓ_∞
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3)
$\ell_\infty \subset X^*$
2)
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