Metric characterization of Linear Operators

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Factoring through Spaces

For $n \geqslant 1$, the binary tree $B_n := \{\emptyset\} \cup_{i=1}^{n} \{0,1\}^i$ is a finite metric space with the shortest path metric

$$d(s,t) = |s| + |t| - 2|u|$$

where u is the nearest common ancestor of s and t.
$B_\infty = \cup_{n \geqslant 1} B_n$ is the infinite binary tree.

Theorem (Bourgain, 1986)

Let X be a Banach space. Then X is not superreflexive $\Leftrightarrow \exists D \geqslant 1$ and maps $f_n \colon B_n \to X$ such that

$$\frac{d(s,t)}{D} \leqslant \|f_n(s) - f_n(t)\| \leqslant d(s,t).$$

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Let $A\colon X \to Y$ be a linear operator between Banach spaces.

Definition

Let $\mathcal{M}$ be a family of metric spaces (M, d). Then $\mathcal{M}$ factors through A if $\exists D \geqslant 1$ s.t. $\forall M \in \mathcal{M}\, \exists f\colon M \to X$

$$\|f(s) - f(t)\| \leqslant d(s,t) \qquad \text{and} \qquad \|Af(s) - Af(t)\| \geqslant \frac{d(s,t)}{D}.$$

Note that

$$\|f(s) - f(t)\| \geqslant \frac{1}{\|A\|}\|Af(s) - Af(t)\| \geqslant \frac{1}{\|A\|D}d(s,t)$$

and

$$\|Af(s) - Af(t)\| \leqslant \|A\|\|f(s) - f(t)\| \leqslant \|A\|d(s,t).$$

Super weakly compact operators

Definition

- Let $A\colon X \to Y$ and $A_1 : X_1 \to Y_1$ be continuous linear operators. Then A_1 is finitely representable in A if $\forall \varepsilon > 0$ and $\forall$ finite-dimensional subspaces $E_1 \subset X_1$, $\exists E \subset X$, isomorphisms $U\colon E_1 \to E$, $V\colon A_1(E_1) \to A(E)$ such that $\|U\|\|U^{-1}\| < 1 + \varepsilon$, $\|V\|\|V^{-1}\| < 1 + \varepsilon$, and

$$
\begin{array}{ccc}
E_1 & \xrightarrow{A_1} & A_1(E_1) \\
\downarrow U & & \downarrow \quad V \\
E & \xrightarrow{A} & A(E)
\end{array}
$$

- $A\colon X \to Y$ is super weakly compact if $A_1 : X_1 \to Y_1$ is weakly compact whenever A_1 is finitely representable in A
- X is super-reflexive if $I\colon X \to X$ is super weakly compact.

Operator versions of the results for spaces

Theorem

A is not super weakly compact $\Leftrightarrow$ *the* D_n*'s factor through A*

Theorem

A is not super weakly compact $\Leftrightarrow$ D_∞ *factors through A*

Uniform Convexity

Definition

X is uniformly convex (UC) if $\forall \varepsilon > 0 \, \exists \delta > 0$ such that $\forall x, y \in B_X$ with $\|x - y\| \geqslant \varepsilon$, then

$$\|\frac{x+y}{2}\| \leqslant 1 - \delta.$$

Theorem (Enflo, '73)

If X is superreflexive then X is uniformly convexifiable.

Ingredients of the proof

Kloeckner's ('14) short self-improvement argument and Beauzamy's operator version of Enflo's renorming theorem.

Theorem (Beauzamy, '76)

$A\colon X \to Y$ is super weakly compact $\Leftrightarrow$ X admits an equivalent norm $|\cdot|$ such that A is uniformly convexifying, i.e. $\forall \varepsilon > 0\, \exists \delta > 0$ such that $\forall x, y \in B_X$

$$\|Ax - Ay\| \geqslant \varepsilon \Rightarrow |\frac{x+y}{2}| \leqslant 1 - \delta.$$

The converse uses James's characterizations of weak compactness.

Theorem (James '72)

$A\colon X \to Y$ is not weakly compact $\Leftrightarrow \exists (x_n) \subset B_X$ and $\theta > 0$ such that (Ax_n) is a basic sequence and

$$\|\sum_{n\in B} Ax_n\| \geqslant \theta |B| \qquad (B \subset \mathbb{N} \text{ finite})$$

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Asymptotic Versions: Property (β) of Rolewicz

Lemma

Suppose X is uniformly convex. If $x, y, z \in B_X$ and $\|y - z\| \geqslant \varepsilon$ then

$$\min(\|x + y\|, \|x + z\|) \leqslant 2 - 2\delta(\varepsilon/2).$$

Proof.

Either $\|x - y\| \geqslant \varepsilon/2$ or $\|x - z\| \geqslant \varepsilon/2$. Suppose the latter. Then

$$\|x + y\| \leqslant 2(1 - \delta(\varepsilon/2)).$$

□

Definition (Kutzarova, '91)

X has property β of Rolewicz if
$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in B_X \, \forall (x_n) \subset B_X$, with $\inf_{m \neq n} \|x_m - x_n\| \geqslant \varepsilon$, we have

$$\inf_{n \geqslant 1} \|x - x_n\| \leqslant 2 - \delta.$$

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Comparison of UC and the β property

For $1 < p < \infty$, $(\sum_{n=1}^{\infty} \ell_\infty^n)_{\ell_p}$ has (β) but is not uniformly convexifiable.

Similarities

- *UC $\Rightarrow$ (β) $\Rightarrow$ reflexive.*
- (β) passes to subspaces and quotients.
- X is (β)-able $\Leftrightarrow$ X^* is (β)-able.
- (β)-ability is inherited by uniform nonlinear quotients (D-Kutzarova-Randrianarivony, '16)
- A reflexive space X is (β)-able $\Leftrightarrow$ X is asymptotically uniformly convexifiable and asymptotically uniformly smoothable.

The referee of a recent paper suggested the term asymptotically superreflexive for (β)-able.

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Szlenk index

Let $K \subset X^*$ be w^*-compact and let $\varepsilon > 0$.

- The Szlenk derivation is defined by

$$s_\varepsilon(K) = K \setminus \cup\{V\colon V\ w^*\text{-open}, \operatorname{diam}(V \cap K) \leqslant \varepsilon\}$$

- Let $s_\varepsilon^0(K) = K$. If ξ is an ordinal, define

$$s_\varepsilon^{\xi+1}(K) = s_\varepsilon(s_\varepsilon^\xi(K)).$$

  If ξ is a limit ordinal, define

$$s_\varepsilon^\xi(K) = \cap_{\zeta<\xi} s_\varepsilon^\zeta(K).$$

-

$$\mathrm{Sz}(K,\varepsilon) = \begin{cases} \min\{\xi\colon s_\varepsilon^\xi(K) = \emptyset\}, & \text{if the set is nonempty} \\ \infty, & \text{otherwise} \end{cases}$$

- $\mathrm{Sz}(K) = \sup_{\varepsilon>0} \mathrm{Sz}(K,\varepsilon)$

Asymptotic Results: Space Results

Finally, the Szlenk index of X is defined as $\mathrm{Sz}(X) := Sz(B_{X^*})$.

Let $T_n := \{\emptyset\} \cup_{i=1}^{n} \mathbb{N}^i$ with the tree metric: infinitely branching tree of depth n.

Theorem (Baudier-Kalton-Lancien, '10)

If $\mathrm{Sz}(X) > \omega$ or $\mathrm{Sz}(X^*) > \omega$ then the T_n's embed with uniform distortion into X and X^*.

Theorem (Baudier-Kalton-Lancien '10)

If X is reflexive, $\mathrm{Sz}(X) = \omega$, and $\mathrm{Sz}(X^*) = \omega$, then the T_n's do not embed with uniform distortion into X.

To extend these results to operators we also extend the following result. This is the asymptotic version of Enflo's renorming theorem

Theorem (D-Kutzarova-Lancien-Randrianarivony, '16)

X admits an equivalent norm with property (β) of Rolewicz $\Leftrightarrow$ X is reflexive, $\mathrm{Sz}(X) = \omega$, and $\mathrm{Sz}(X^*) = \omega$.

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Definition

Let $A\colon X \to Y$ be a linear operator. The Szlenk index of A is defined as $\mathrm{Sz}(A) = Sz(A^*(B_{Y^*}))$.

Theorem

If $\mathrm{Sz}(A) > \omega$ or $\mathrm{Sz}(A^*) > \omega$ then the T_n's factor through A and A^*.

Definition

Let $A\colon X \to Y$.

- A has property (β) if $\forall \varepsilon > 0\, \exists \delta > 0\, \forall x \in B_X\, \forall (x_n) \subset B_X$, with $\inf_{m \neq n} \|Ax_m - Ax_n\| \geqslant \varepsilon$, we have
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Remark

This is the asymptotic version of Beauzamy's renorming theorem

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The proof uses the self-improvement argument as in [Baudier-Zhang, '15].

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Remark

Our renorming results use techniques and results from [Lancien-Prochazka-Raja, ’15] on higher order asymptotic uniform convexity.