①

Strong algebraization of fixed point properties

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§ 0. Outline

$\mathcal{X}$: a class of Banach spaces

$(F_{\mathcal{X}})$: the fixed point property w.r.t. $\mathcal{X}$

One approach to $(F_{\mathcal{X}})$:

"the Part & the Whole" method

Step 1. the Part (rel $(F_{\mathcal{X}})$)

Main Topic today ⇒ Step 2. SYNTHESIS step

(the Part $\Longrightarrow$ the Whole)

③

One strategy for SYNTHESIS:

$\boxed{\text{ALGEBRAIZATION}}$

(Y. SHALOM, 1999 Publ. IHÉS; '06 ICM)

OK for $\mathbb{Z}$ with "UNBOUNDED WILDNESS"

Always imposED

"<u>BOUNDED GENARATION</u> assumption"

(BG)

<u>Def</u> G a grp; $M_1, \dots, M_\ell \leq G$ subgrps

$M_1, \dots, M_\ell$ (BG) $G \overset{\text{def}}{\iff} \exists N \in \mathbb{N}$ s.t. $\forall g \in G$

$g = h_1 \cdots h_N \quad \exists h_j \in \bigcup_{i=1}^{\ell} M_i$

④ Main Result (M, '15+)

Obtain a new algebraization that is FREE FROM (BG)

"STRONG ALGEBRAIZATION"

Cor (M, '15+) R: $\ni 1$, f.g. associative ring

(1) $E(n, R)$ ⇦ elementary group for $\forall n \geq 4$ has $(F_{\mathcal{B}^{NC}_{L_p}})$ for $\forall p \in (1, \infty)$

(2) [ERSHOV–JAIKIN '10 Invent. Math] $E(n, R)$ $\forall n \geq 3$ has (T) $(\Leftrightarrow (F_{\mathcal{B}_{L_2}}))$ ⇦ "SA" gives simpler proof

⑤ $\S 1$ $(F\mathcal{H})$

G a finitely generated group
$\vee\!/\!/$
M a subgroup

Def

(1) G has $(F\mathcal{H})$

$\underset{\text{def}}{\Longleftrightarrow}$ $\forall X \in \mathcal{H}$, $\forall \alpha : G \curvearrowright X$ ⟸ affine isometric action

$\left(\|\alpha(g)\cdot x - \alpha(g)\cdot y\| = \|x - y\|\right)$

$\left\{\begin{matrix}\alpha(G)\text{-}\\ \text{fixed}\\ \text{pt}\end{matrix}\right\} \rightarrow X^{\alpha(G)} \neq \emptyset$

(2) $M \leq G$ has rel $(F\mathcal{H})$

$\underset{\text{def}}{\Longleftrightarrow}$ $\forall X \in \mathcal{H}$, $\forall \alpha : G \curvearrowright X$

$X^{\alpha(M)} \neq \emptyset$

⑥

Basic ex. of $\mathfrak{X}$: $p \in (1, \infty)$ fixed

$$\mathcal{B}_{NCL_p} := \{ L_p(\mathcal{N}) \mid \mathcal{N} : \text{a vN alg} \}$$
$$\cap\!|$$
$$\mathcal{B}_{uc} := \{ \text{uniformly convex} \}$$
$$\cap\!|$$
$$\mathcal{B}_{sr} := \{ \text{superreflexive} \}$$
$$\cap\!|$$
$$\mathcal{B}_{type>1} := \{ \text{(Rademacher) type} > 1 \}$$

Rem $(F_{\mathcal{B}_{NCL_2}}) \, (= (F_{Hilbert})) \iff$ KAZHDAN's property (T)

Ex

⊙ $SL(n, \mathbb{Z})$, $\forall n \geq 3$ has

$(F_{\mathcal{B}_{ncL_p}})$ for $\forall p \in (1, \infty)$

[BADER-FURMAN-GELANDER-MONOD + OLIVIER]

⊙ $SL(n, \mathbb{F}_3[t])$, $\forall n \geq 3$ has

$(F_{\mathcal{B}_{type>1}})$ [V. LAFFORGUE]

⊙ $R \ni 1$, f.g. & associative ring (possibly non-commutative)

$$E(n, R) := \left\langle e_{ij}(r) \;\middle|\; \begin{array}{l} i \neq j \in [n] = \{1, \ldots, n\} \\ r \in R \end{array} \right\rangle, \quad \forall n \geq 3$$

"elementary" group

$$e_{ij}(r) = \begin{array}{c c} & \begin{array}{ccccc} & & & j & \end{array} \\ i & \begin{bmatrix} 1 & & & & \\ & 1 & \cdots & r & \cdots \\ & & \ddots & \vdots & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array}$$

has $(T) \Leftrightarrow (F_{\mathcal{B}_{ncL_2}})$

[ERSHOV-JAIKIN '10 Invent.]

Who cares?

(1) G (∞ group) $(F_{\mathcal{B}_{\mathrm{NCL}_p}})$ for $\forall p \in (1, \infty)$

$\Longrightarrow$ G "FAR FROM" GROMOV-hyperbolic

BOURDON-PAJOT...

(2) $\mathcal{X} \subseteq \mathcal{B}_{\mathrm{sr}}$ stable under

- ultraproduct (w.r.t. $\exists$ free ultrafilter)
- ℓ_q-sum for $\exists q \in (1, \infty)$

Then G $(F_{\mathcal{X}})$

$\Longleftrightarrow$ G robust $(T_{\mathcal{X}})$

OPPENHEIM + de la SALLE

$\rightsquigarrow$ finite grp quots "Geometric" robust $(T_{\mathcal{X}})$

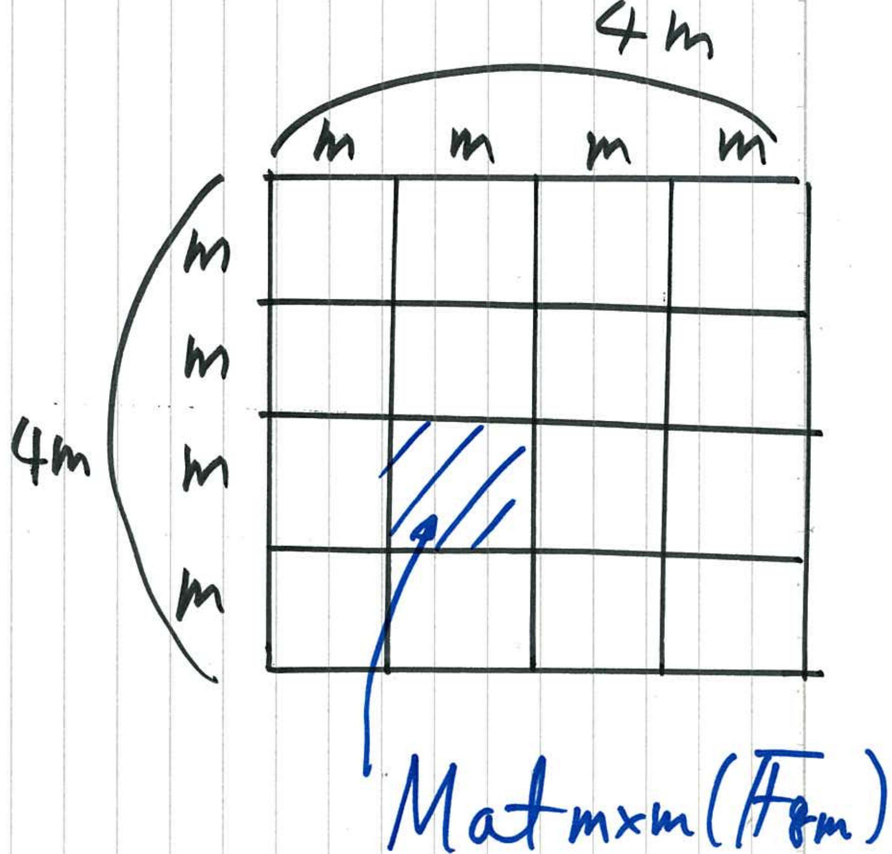
$\rightsquigarrow$ ——//—— $\mathcal{X}$-expander graphs

⑨ ex

$SL(4m, \mathbb{F}_{q_m})$ $(q_1, q_2, \cdots$: primes$)$

ex

$SL(4m, \mathbb{F}_{q^m})$



(11) ex

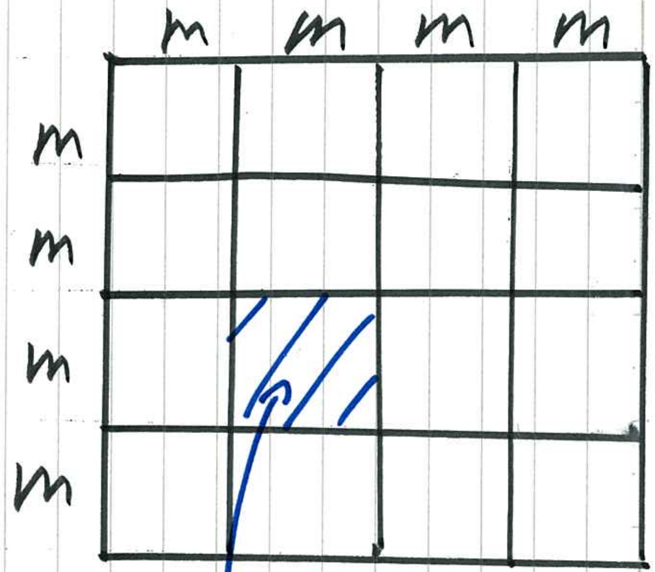
$SL(4m, \mathbb{F}_{q^m})$

$\cong$

$E(4, \underbrace{Mat_{m\times m}(\mathbb{F}_{q^m})}_{})$

for $\forall m$, generated by

$$I_m, \quad x_m = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}, \quad y_m = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}$$



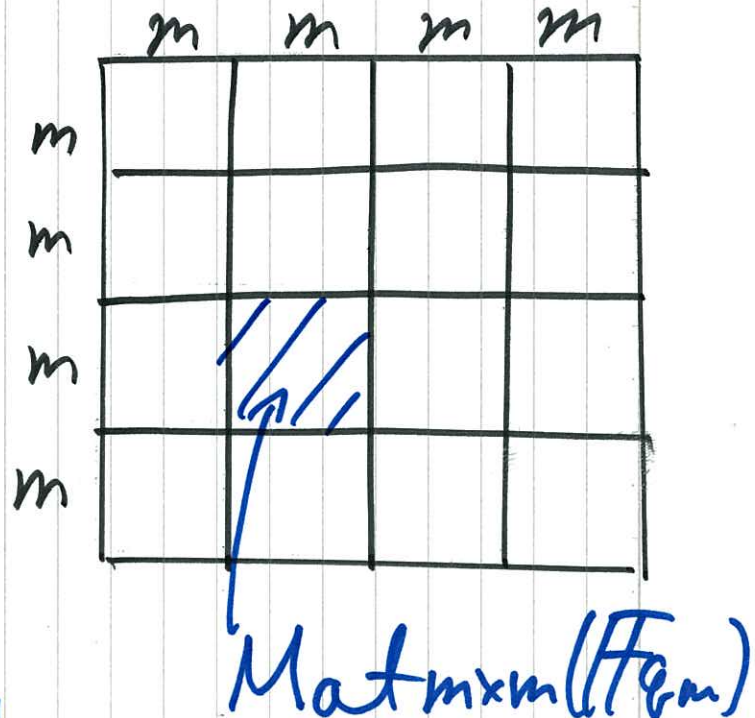
$Mat_{m\times m}(\mathbb{F}_{q^m})$

⑫ ex 1

$$E(4, \mathbb{Z}\langle x, y\rangle)$$

$$\downarrow$$

$$SL(4m, \mathbb{F}_{q^m}) \cong E(4, \mathrm{Mat}_{m\times m}(\mathbb{F}_{q^m}))$$



⊙ E-J's thm ((T)) $\rightsquigarrow$ $\underbrace{\{SL(4m, \mathbb{F}_{q^m})\} \text{ expanders}}$

⊙ Cor[M] $((\mathrm{F}_{B_n C_4})\ \forall p) \rightsquigarrow \{\ \nrightarrow\ \}$ $\underline{\underline{\text{Geometric robust}}}$ $\underline{\underline{(T_{B_n C_4})\ \forall p}}$.

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§2. the Part & the Whole strategy

Step 1: Show $\mathrm{rel}(F_{Z})$ for $G \geqslant \underbrace{M_1 \cdots M_\ell}_{\text{"good" subgrps}}$

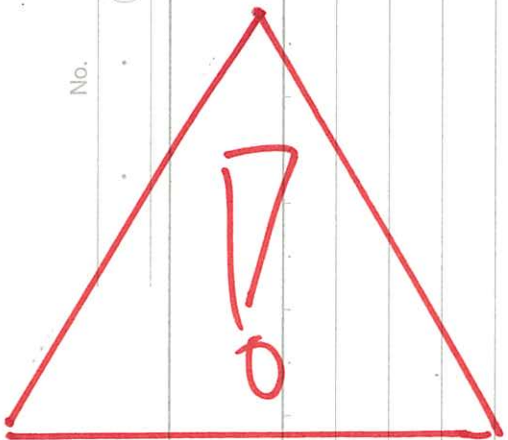
"the Part"

Step 2: (SYNTHESIS step)

Synthesize $\mathrm{rel}(F_{Z})$ into the Whole (F_{Z})

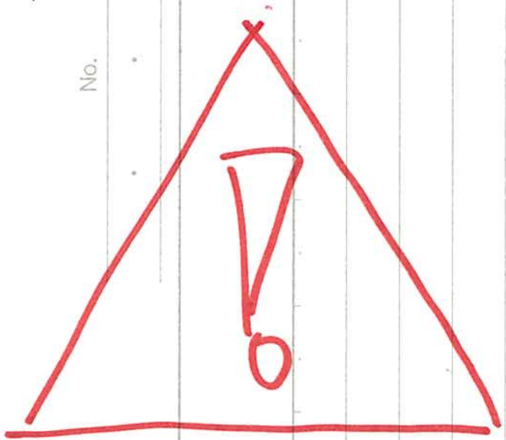
2 streams:

- algebraization [SHALOM] ⇐ today
- $\mathfrak{C}$-orthogonality [E-J, OPPENHEIM, LAVY, …]



$\alpha : G \curvearrowright X$, $M_1, \dots, M_\ell \le G$

$$\left.\begin{aligned} &\odot\ \langle M_1, \dots, M_\ell \rangle = G \\ &\odot\ X^{\alpha(M_i)} \neq \emptyset \quad \forall i \end{aligned}\right\} \Longrightarrow X^{\alpha(G)} \neq \emptyset$$



$\alpha : G \curvearrowright X$, $M_1, \dots, M_\ell \leq G$

$$\left.\begin{array}{l} \odot\ \langle M_1, \dots, M_\ell \rangle = G \\ \\ \odot\ X^{\alpha(M_i)} \neq \emptyset \\ \qquad \forall i \end{array}\right\} \cancel{\Rightarrow}\ X^{\alpha(G)} \neq \emptyset$$

this is the GOAL...

NOT true (unless $\left(\bigcap_{1 \leq i \leq \ell} X^{\alpha(M_i)} \neq \emptyset \dots \right)$

(16)

§ 3. SHALOM's first algebraization

Thm (SHALOM 1999, Publ. IHÉS)

X reflexive

f.g. grp $\to G \geq M_1 \ldots M_\ell$

axiom $(G \geq) M_1 \ldots M_\ell$ (BG) G

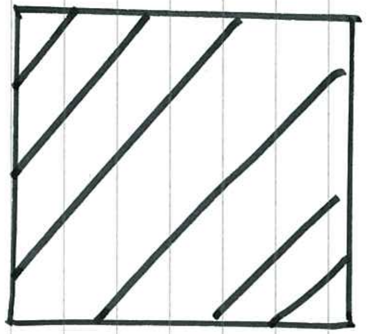
$\left(\begin{array}{l} \exists N \in \mathbb{N} \text{ s.t.} \\ \forall g \in G \\ g = h_1 \cdots h_N \\ \exists h_j \in \bigcup_{i=1}^{\ell} M_i \end{array} \right)$

Then: $\boxed{\forall i \quad G \geq M_i \ \text{rel}(F_X) \Longrightarrow G \ (F_X)}$

Point X ref. $\rightsquigarrow$ $X^{\alpha(M)} \neq \emptyset$

$\alpha : M \curvearrowright X$ $\iff \exists x \in X$ s.t. $\alpha(M) \cdot x$ bounded

$\iff \forall y \in X$ s.t. $\alpha(M) \cdot y$ bounded (ex) 17 $R \ni 1$ a f.g. (associative) ring, $n \geqslant 3$

$G = E(n, R) =$ 

$M = \begin{pmatrix} I_{n-1} & \\ 0 \cdots 0 & 1 \end{pmatrix} \quad (\simeq (R^{n-1}, +))$

$L = \begin{pmatrix} I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ & 1 \end{pmatrix} \quad (\simeq (R^{n-1}, +))$

Conj (M.)
$\forall n \geqslant 4$. rel(FIB_{sr})

Fact

(1) [KASSABOV + M. + OLIVIER]

$\forall n \geqslant 4$

$G \geqslant M$, $G \geqslant L$ $\Big\}$ rel(FIB_{ncL_p}) for $\forall p \in (1, \infty)$

(2) [KASSABOV]

$\forall n \geqslant 3$

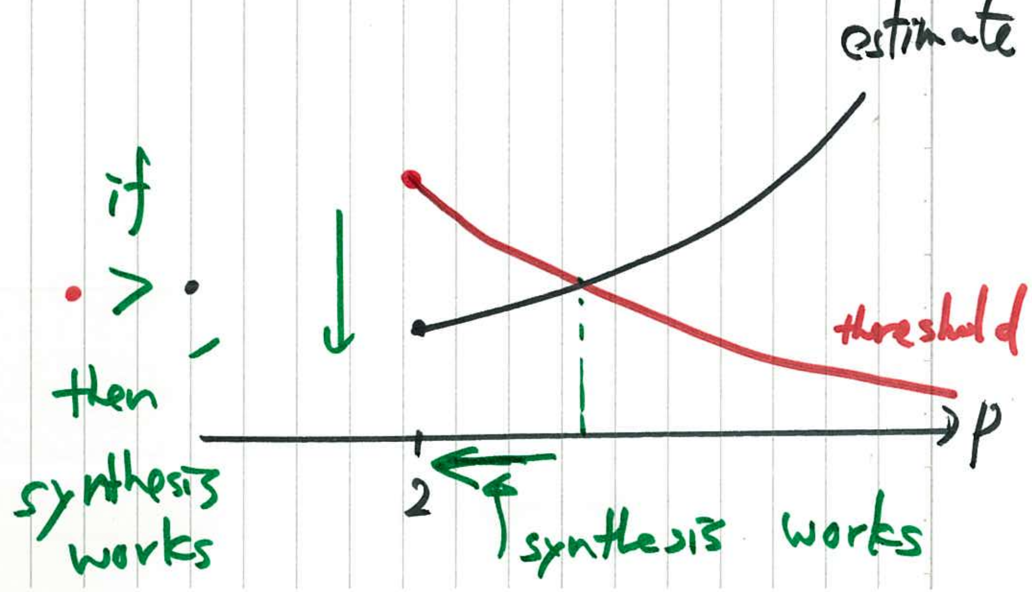
〃 〃, 〃 〃 $\Big\}$ rel(T) $=$ rel(FIB_{ncL_2})

(18)

⚠ (BG) is UNKNOWN (unless $R=\mathbb{Z}$ ...)

[ERSHOV-JAIKIN '10, Invent. Math.]
used "ε-orthogonality" to synthesize (2).

BUT.. this method doesN'T
work to synthesize (1)
"for all $p\in(1,\infty)$"



§4. Main Result

(2-subgrp version)

Main Thm (M '15+, strong algebraization)

G a f.g. grp

$\mathcal{K} \subseteq \mathcal{B}sr$, stable under ultraproducts

$G \geqslant M, L$; $\mathrm{Aut}(G) \geqslant \Pi$

Axioms (i) $\langle M, L \rangle = G$

(ii) $|\Pi| < \infty$

(Game) We can win the (GAME) for (G, M, L, Π)

Then $\boxed{\begin{matrix} G \geqslant M \\ G \geqslant L \end{matrix} \ \mathrm{rel}\,(F_{\mathcal{K}}) \Longrightarrow G\,(F_{\mathcal{K}})}$

the (GAME) (G, M, L, Π) given (fixed)

2 "characters" H_1, H_2 : 2 subgroups of G (vary)

- First,

H_1	H_2
M	L

- We can enlarge H_1 & H_2 by admissible MOVES (I), (II_{id}), $(\mathrm{II}_{(12)})$
- We win if (within finite steps of moves), we can set G either for H_1 or for H_2.

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MOVES : (I), (II_{id}), ($\mathrm{II}_{(12)}$)

(I) [cf. SHALOM '06, ICM proceedings]

Choose $P \subseteq G$ s.t.

$$\left[\ \forall g \in P \quad \begin{array}{l} g H_1 g^{-1} \geq M \\ g H_2 g^{-1} \geq L \end{array}\right.$$

<u>Then</u>: enlarge as

H_1	H_2
$\langle H_1, P\rangle$	$\langle H_2, P\rangle$

(22)

(II_{id}) Choose $\Lambda \subseteq \Pi \cdot \mathrm{Inn}(G) \ (\leq \mathrm{Aut}(G))$

s.t. $\forall \phi \in \Lambda \quad \phi^{-1}(H_1) \geq M$

$\phi^{-1}(H_2) \geq L$

Then: enlarge as

H_1	H_2
$\langle H_1, \bigcup_{\phi\in\Lambda} \phi(H_1) \rangle$	$\langle H_2, \bigcup_{\phi\in\Lambda} \phi(H_2) \rangle$

$(\mathrm{II}_{(12)})$ Choose $\Lambda \subseteq \Pi \cdot \mathrm{Inn}(G)$

s.t. $\forall \phi \in \Lambda \quad \phi^{-1}(H_1) \geq L$

$\phi^{-1}(H_2) \geq M$

Then: enlarge as

H_1	H_2
$\langle H_1, \bigcup_{\phi\in\Lambda} \phi(H_2) \rangle$	$\langle H_2, \bigcup_{\phi\in\Lambda} \phi(H_1) \rangle$

23 ex $[\text{Main Thm} \Rightarrow \text{Cor (M)}]$

$$\begin{pmatrix} \ast \end{pmatrix} \leftarrow G\,(=E(n,R)) \qquad M = \left(\begin{array}{c|c} I_{n-1} & \ast \\ \hline 0 \cdots 0 & 1 \end{array}\right) \qquad L = \left(\begin{array}{c|c} I_{n-1} & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ \hline \ast & 1 \end{array}\right)$$

$$\Pi = \langle \tau \rangle = \{ id_G, \tau \} \qquad \tau : 1^{st} \rightleftarrows n^{th}$$

(GAME)

H_1	H_2
M	L

$\overset{\rightsquigarrow}{(I)}$

$$P = \left(\begin{array}{c|c} \ast & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right) \quad (\simeq E(n-1,R))$$

(check $\forall g \in P \quad gMg^{-1} \geq M$, $gLg^{-1} \geq L$)

$$\left(\begin{array}{c|c} \ast & \ast \\ \hline 0 \cdots 0 & 1 \end{array}\right) = \begin{array}{c|c} H_1 & H_2 \\ \hline \langle M, P \rangle & \langle L, P \rangle \end{array} = \left(\begin{array}{c|c} \ast & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ \hline \ast & 1 \end{array}\right)$$

$\longrightarrow$ $(\mathrm{II}_{(12)})$

$$\Delta = \{\psi\} \quad (\leq \Pi\text{-}\mathrm{Inn}(G))$$

$$\odot \quad \psi^{-1}(H_1) = \psi(H_1) = \begin{pmatrix} 1 & 0 \cdots 0 \\ * & * \end{pmatrix}$$

$$H_1 = \begin{pmatrix} * & * \\ 0 \cdots 0 & 1 \end{pmatrix}$$

$\left(\psi : \; 1^{st} \rightleftarrows n^{th} \right)$

$$\odot \quad \psi^{-1}(H_2) = \psi(H_2) = \begin{pmatrix} 1 & * \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & * \end{pmatrix}$$

$$H_2 = \begin{pmatrix} * & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ * & 1 \end{pmatrix}$$

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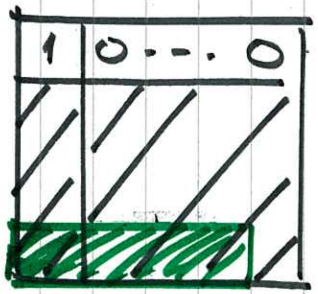
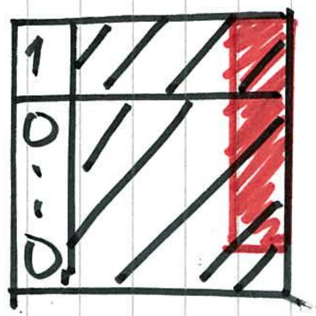
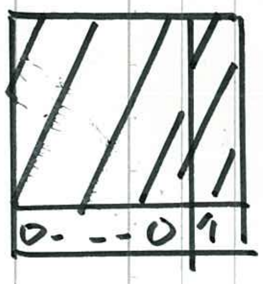
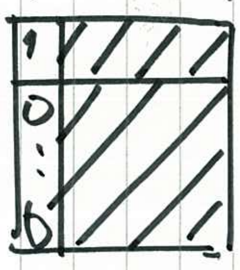
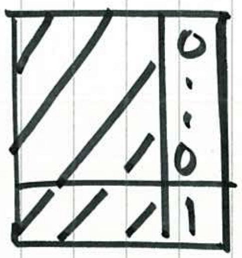
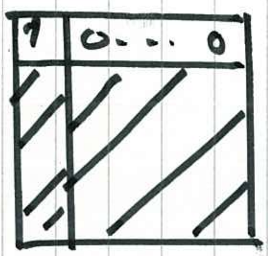
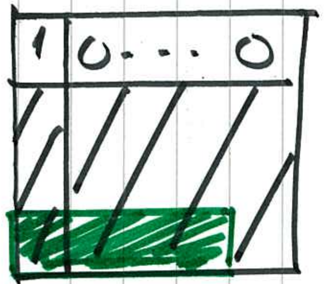
$\longrightarrow$ $(\mathrm{II}_{(12)})$ $\Lambda = \{\psi\}$

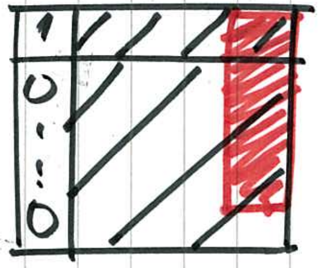
$\odot$ $\psi^{-1}(H_1) = \psi(H_1) =$  $\geq L$

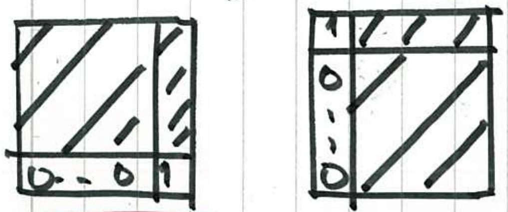
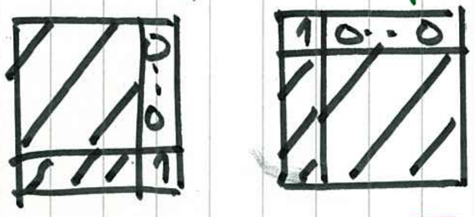
$\odot$ $\psi^{-1}(H_2) = \psi(H_2) =$  $\geq M$

H_1	H_2
$\langle H_1, \psi(H_2) \rangle$	$\langle H_2, \psi(H_1) \rangle$
 	 

$\xrightarrow{}$ $(\mathrm{II}_{(1\,2)})$ $A = \{\psi\}$

$\odot$ $\psi^{-1}(H_1) = \psi(H_1) =$  $\geq L$

$\odot$ $\psi^{-1}(H_2) = \psi(H_2) =$  $\geq M$

H_1	H_2
$\langle H_1, \psi(H_2)\rangle$  $= G$	$\langle H_2, \psi(H_1)\rangle$  $= G$

((i) $\langle M\, L\rangle = G$

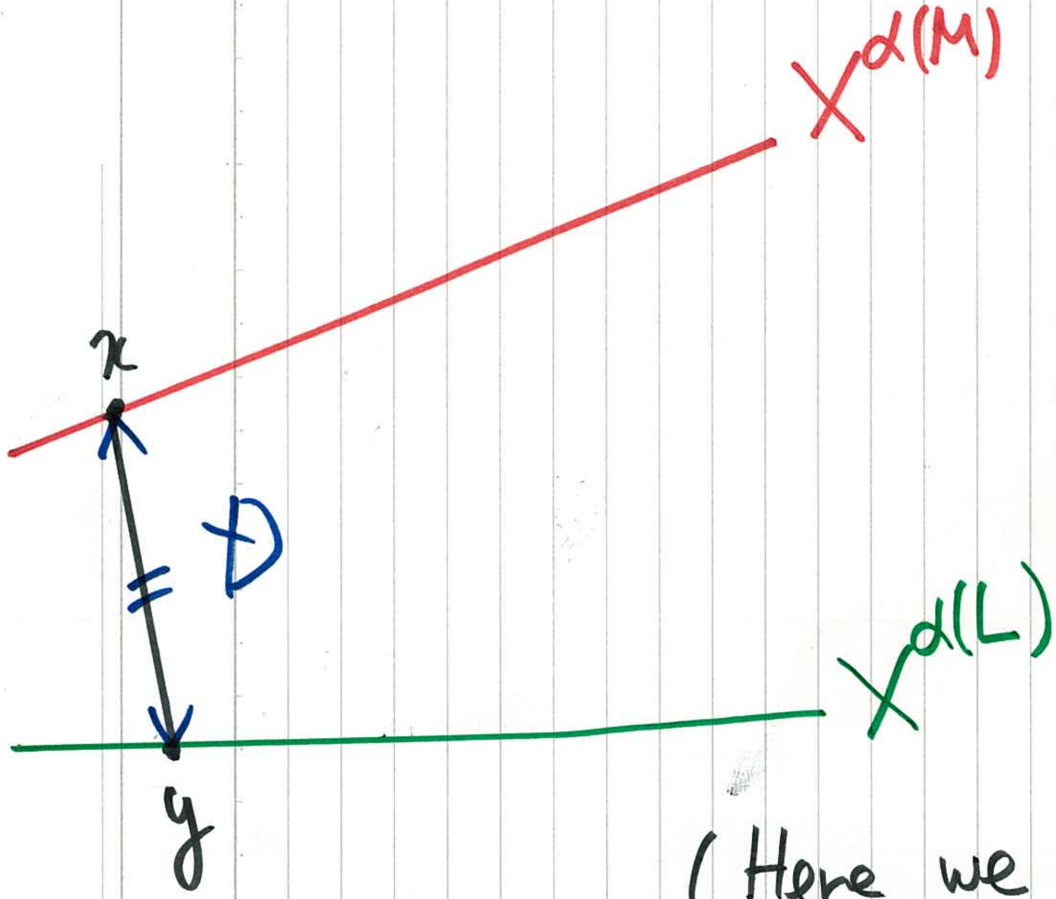
(ii) $|\Pi| < \infty$

are also OK.)

§5. On the proof of MT (WMA $\mathcal{X} \subseteq \mathcal{B}_{\mathrm{uc}}$)

By contradiction: Assume G NOT $(F_{\mathcal{X}})$

Then $\exists X \in \mathcal{X}$, $\exists \alpha : G \curvearrowright X$ s.t.



① $X^{\alpha(G)} = \emptyset$

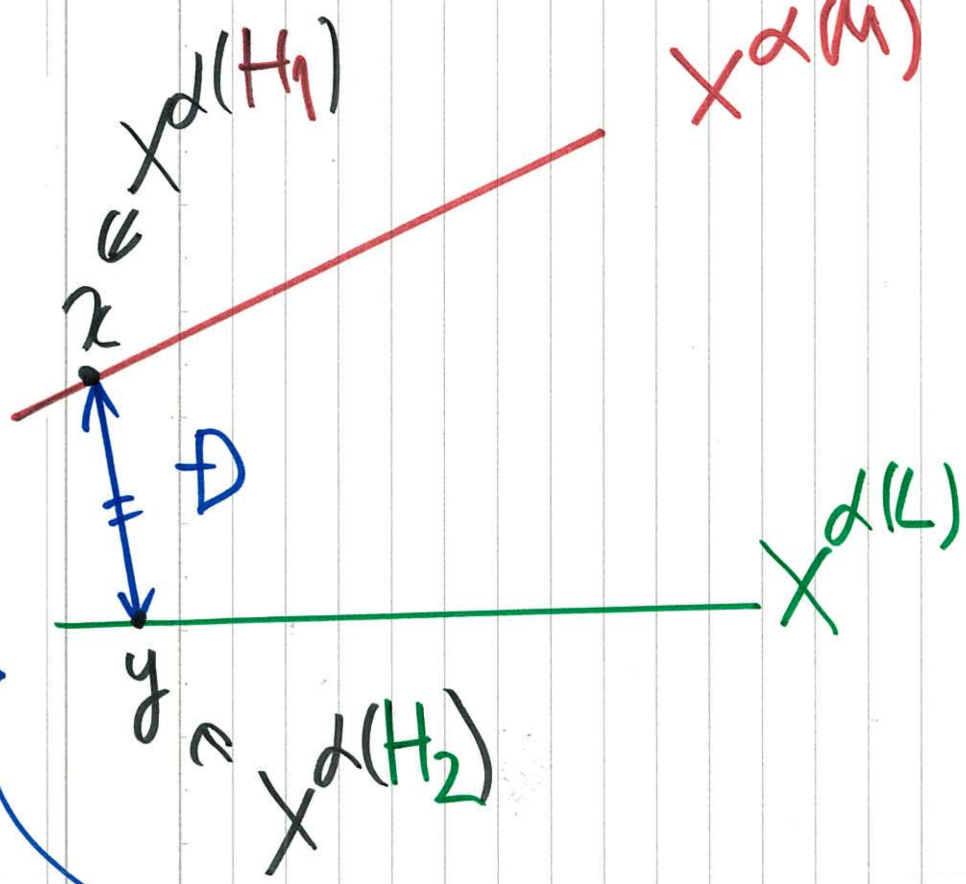
and

② $\exists x \in X^{\alpha(M)}$, $\exists y \in X^{\alpha(L)}$ that realize

$$\mathrm{dist}(X^{\alpha(M)}, X^{\alpha(L)}) =: D$$

(Here we take ultraproducts twice!)

(28) <u>Self-improvement argument</u>:



For all realizers

$$(x, y)$$

$$x \in X^{\alpha(M)}, \quad y \in X^{\alpha(L)},$$

(under more assumptions,)

after each steps of moves

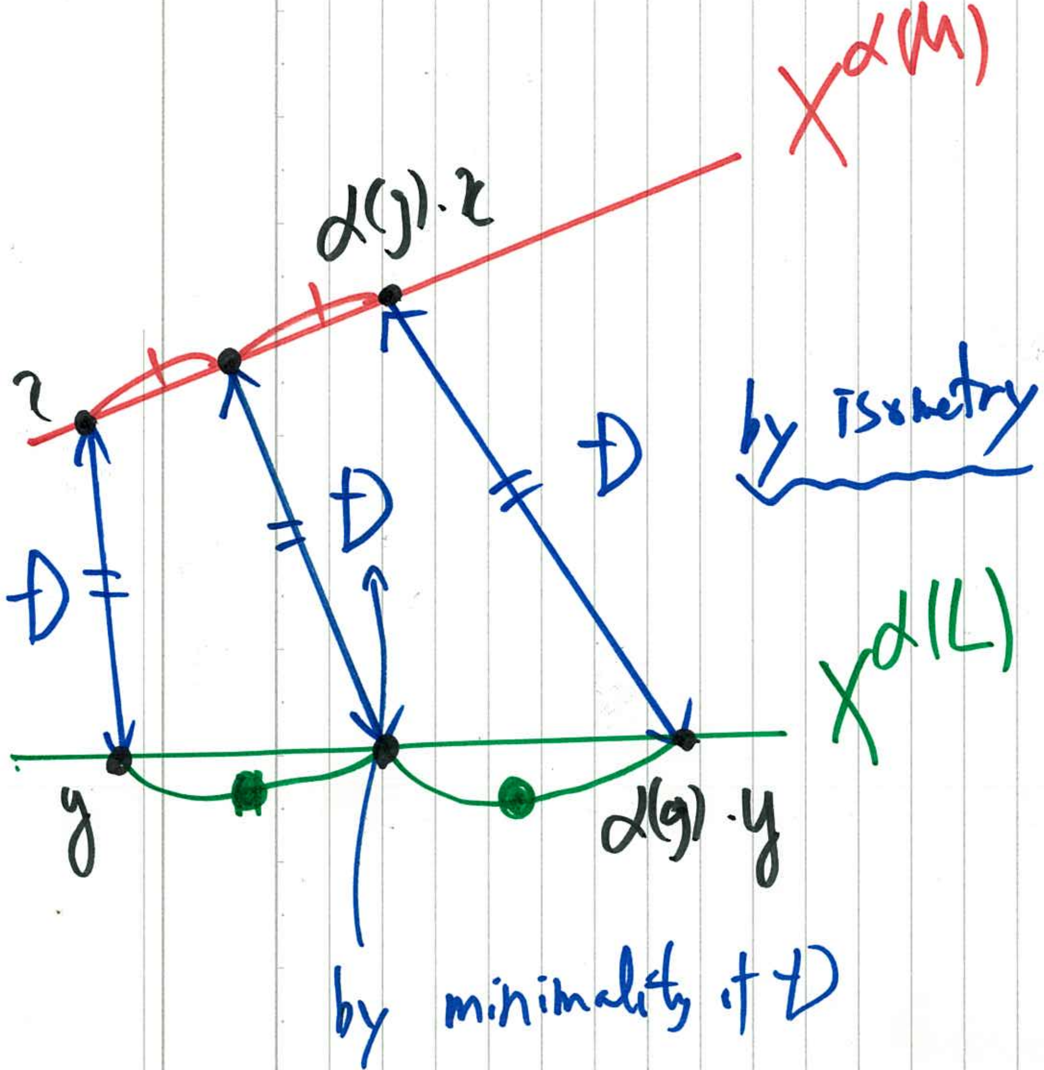
$$\begin{cases} x \in X^{\alpha(H_1)} \\ y \in X^{\alpha(H_2)} \end{cases}$$

[$\xrightarrow{\text{(GAME)}}$ contradicts (1) ! ■]

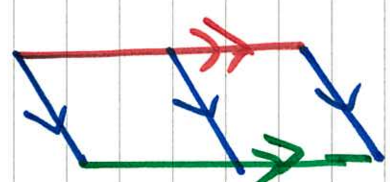
Proof for (I): Pick $\forall g \in \Gamma$.

conditions

$$\begin{cases} g H_1 g^{-1} \geq M \\ g H_2 g^{-1} \geq L \end{cases} \rightsquigarrow \begin{cases} \alpha(g)\cdot x \in X^{\alpha(M)} \\ \alpha(g)\cdot y \in X^{\alpha(L)} \end{cases}$$



by (strict) convexity of X



some splitting of X

$$\forall g \in \Gamma \quad \begin{cases} \alpha(g)\cdot x = x \\ \alpha(g)\cdot y = y \end{cases}$$

//

(30)

§6. Questions to Banach space geometers

(Q1) Can extend beyond $\mathcal{B}_{SL}$ (e.g. $\mathcal{B}_{\mathrm{type}>1}$)?

(Q2) Can avoid using ultraproducts to obtain minimizer (x, g)?

(Q3) Can extend to $\mathcal{B}_{NCL_1}$ ($\Leftarrow$ L-embedded in the sense of BADER-GELANDER-MONOD)

(cf. For $G = E(n, R)$, rel$(F\mathcal{B}_{NCL_1})$ for $\forall n \geq 4$ is OK.)

(Q4) Non-isometric affine actions?

THANK YOU!

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