Essential Circles and their Applications

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1. Discrete Homotopy Primer
2. Homotopy Critical Values and Essential Circles
3. A simple counting argument
4. Applications: Generalizations of Theorems of Gromov, Anderson, Shen-Wei

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4. Applications: Generalizations of Theorems of Gromov, Anderson, Shen-Wei
5. Extension to Topological Invariants
6. Preliminary Comments about Resistance Metrics

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- 2016 (In Preparation): Topological versions of these concepts

Discrete Homotopies in a Metric Space

Let X be a metric space.

Definition

For $\varepsilon > 0$, an ε-chain is a finite sequence $\{x_0, \ldots, x_n\}$ such that for all i, $d(x_i, x_{i+1}) < \varepsilon$. For any ε-chain $\alpha = \{x_0, \ldots, x_n\}$, we define its *length* by

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Definition

An ε-homotopy consists of a finite sequence $\langle \gamma_0, \ldots, \gamma_n \rangle$ of ε-chains, where each γ_i differs from its predecessor by a "basic move": adding or removing a *single* point, always leaving the endpoints fixed.

Epsilon-Covers

Definition

Fixing a basepoint $*$, X_ε is defined to be the set of all ε-homotopy equivalence classes of ε-chains starting at $*$, and $\phi_\varepsilon : X_\varepsilon \longrightarrow X$ is the endpoint map. Equivalence classes are denoted by $[\alpha]_\varepsilon$.

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Definition

The group $\pi_\varepsilon(X)$ is the subset of X_ε consisting of classes of ε-loops starting and ending at $*$ with operation induced by concatenation, i.e., $[\alpha]_\varepsilon * [\beta]_\varepsilon = [\alpha * \beta]_\varepsilon$. We denote the reversal of a chain α by $\overline{\alpha}$. As expected, for $[\alpha]_\varepsilon \in \pi_\varepsilon(X)$, $([\alpha]_\varepsilon)^{-1} = [\overline{\alpha}]_\varepsilon$, and the identity is $[*]_\varepsilon$.

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- $\pi_\varepsilon(X)$ acts as isometries on X_ε
- $\phi_\varepsilon : X_\varepsilon \to X$ is an isometry from any $\frac{\varepsilon}{2}$-ball onto its image

Additional Properties for Compact Geodesic Spaces

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- If X is semi-locally simply connected, then $\phi_\varepsilon : X_\varepsilon \to X$ is the traditional universal covering map for all small enough ε
- For a Riemannian Manifold X, the metric on X_ε is the traditional lifted metric

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- There is a natural function Λ taking the homotopy class of any path to the ε-homotopy class of a subdivision ε-chain
- Restricting Λ to the fundamental group at any base point yields a homomorphism $\pi_1(X) \to \pi_\varepsilon(X)$, which is surjective when X_ε is connected

Homotopy Critical Values

Definition

An ε-loop λ in a metric space X is called ε-critical if λ is not ε-null, but is δ-null for all $\delta > \varepsilon$. When an ε-critical ε-loop exists, ε is called a homotopy critical value; the collection of these values is called the Homotopy Critical Spectrum.

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- For geodesic spaces, this spectrum is discrete in $(0, \infty)$.
- Homotopy critical values are precisely the values of ε such that the covers X_ε change equivalence type.

Two Basic Examples

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- The 2-torus has two homotopy critical values or one of multiplicity 2

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- There is a natural equivalence of essential circles via "free" discrete homotopies.
- Any three equally spaced points on an essential ε-circle are called an essential ε-triad
- Essential triads are the discrete analog (logically equivalent) to essential circles

Essential Circles and the Homotopy Critical Spectrum

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If X is a compact geodesic space then $\varepsilon > 0$ is a homotopy critical value of X if and only if X contains an essential ε-circle (equivalently an essential ε-triad).

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Lemma

Suppose $T = \{x_0, x_1, x_2\}$ is an essential ε-triad in a geodesic space X and $T' = \{x_0', x_1', x_2'\}$ is any set of three points such that $d(x_i, x_i') < \frac{\varepsilon}{3}$ for all i. If T' is an essential triad then T' is an ε-triad equivalent to T.

Controlled Discreteness of the Homotopy Critical Spectrum

Corollary

If X can be covered by N open $\frac{a}{3}$-balls then there are at most $\binom{N}{3}$ non-equivalent essential triads (equivalently, homotopy critical values counted with multiplicity) that are ε-triads for some $\varepsilon \geq a$.

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- Letting $C(X, \frac{a}{3})$ be the minimum number of $a/3$ balls needed to cover X, recall that by Gromov's Precompactness Criterion $C(X, \frac{a}{3})$ has a uniform lower bound $C_{\mathcal{X}}(\frac{a}{3})$ in any Gromov-Hausdorff precompact class $\mathcal{X}$ of spaces.

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- In particular, there are at most $\binom{C_{\mathcal{X}}(\frac{a}{3})}{3}$ homotopy critical values $\geq a$ for any space X in $\mathcal{X}$

Gromov's Basis Theorem

Theorem

(Gromov) If X is a Riemannian manifold of diameter D,then $\pi_1(X)$ has a set of generators $g_1, ..., g_k$ of length at most $2D$ and relations of the form $g_i g_m = g_j$.

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- Essentially the same argument holds if X is a semilocally simply connected, compact geodesic space
- Advantage: to prove finiteness of fundamental groups one needs only bound the number k of possible generators in some class of spaces, since then there are at most 3^k possible relators, hence at most 2^{3^k} different groups.

Discrete, Quantitative version of Gromov's Theorem

Theorem

Let X be a compact geodesic space of diameter D, and $\varepsilon > 0$. Then $\pi_\varepsilon(X)$ has a finite set of generators $[\gamma_1]_\varepsilon, \ldots, [\gamma_k]_\varepsilon$ and relators of the form $[\gamma_i]_\varepsilon[\gamma_j]_\varepsilon = [\gamma_m]_\varepsilon$ such that $L(\gamma_i) \leq 2(D + \varepsilon)$ for all i

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$$k \leq C\left(X, \frac{\varepsilon}{4}\right)^{\frac{8(D+\varepsilon)}{\varepsilon}}.$$

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- The proof involves constructing a finite 2-D symplicial complex with edge group equal to $\pi_\varepsilon(X)$

What about small loops?

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Theorem

With the previous hypotheses and if for $0 < \delta < \varepsilon$ there are at most M distinct non-trivial elements $[\alpha]_\delta \in \pi_\delta(X)$ such that $|[\alpha]_\delta| < \varepsilon$ then for the number k of generators of $\pi_\delta(X)$ we have

$$k \leq M \left[\frac{8\left(D+\varepsilon\right)}{\varepsilon}\right] \left[C\left(X, \frac{\varepsilon}{4}\right)\right]^{\frac{8(D+\varepsilon)}{\varepsilon}}$$

Final Quantitative Extension of Gromov's Theorem

Hitting the previous theorem with the function Λ, and letting $\Gamma(X, \varepsilon)$ be the maximum number of elements of $\pi_1(X, *)$ of length $\leq \varepsilon$ for any basepoint $*$, we obtain:

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$$k \leq \frac{8(D+\varepsilon)}{\varepsilon} \cdot \Gamma(X,\varepsilon) \cdot C\left(X, \frac{\varepsilon}{4}\right)^{\frac{8(D+\varepsilon)}{\varepsilon}}.$$

Generalizing Anderson, Shen-Wei Finiteness

Corollary

Let $\mathcal{X}$ be any Gromov-Hausdorff precompact class of semilocally simply connected compact geodesic spaces. If there are numbers $\varepsilon > 0$ and N such that for every $X \in \mathcal{X}$, $\Gamma(X, \varepsilon) \leq N$, then there are finitely many possible fundamental groups for spaces in $\mathcal{X}$.

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- Both earlier arguments use, in essential ways, Gromov-Hausdorff precompactness of the universal covers, which may not be true in the present generality.

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- For example, there are flat tori with one or two homotopy critical values depending on the metric
- To get topological versions we need to consider uniform spaces
- Compact topological spaces have a unique uniform structure compatible with the topology
- So properties that depend only on the uniform structure are topological invariants

Entourages and Uniform Spaces

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- The collection of entourages also satisfies a property generalizing the triangle inequality
- Given an entourage E, the E-ball centered at x is $B(x, E) := \{y : (x, y) \in E\}$

Peano Continua and Geodesic Metrics

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- By a theorem of Bing-Moise, any Peano continuum admits a geodesic metric.
- In a geodesic space, if $d(x, y) < \varepsilon$ then x and y lie in the same connected component of $B(x, \varepsilon) \cap B(y, \varepsilon)$ (in fact any minimizing geodesic joining x and y lies in $B(x, \varepsilon) \cap B(y, \varepsilon)$.

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- In the language of uniform spaces this translates as: If $(x, y) \in E_\varepsilon$ then x and y lie in the same connected component of $B(x, E_\varepsilon) \cap B(y, E_\varepsilon)$

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- By the Bing-Moise Theorem, every Peano continuum has a uniform space basis of centourages (namely the metric entourages of any geodesic metric)
- In a compact space, any property that depends only on the collection of centourages is a topological invariant

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- One similarly constructs a space X_E and endpoint mapping $\phi_E : X_E \to X$, where X_E has a natural uniform structure (but may not be compact!)
- With sufficient connectivity properties, in particular if E is a centourage, X_E is connected and ϕ_E is a regular covering map with deck group $\pi_E(X)$ consisting of all E-homotopy classes of E-loops. We call X_E a centourage cover.

Centourage Covers

- In a uniform space, replace the statement "$d(x, y) < \varepsilon$" by "$(x, y) \in E$"
- This generalizes ε-chains and ε-homotopies to E-chains and E-homotopies
- One similarly constructs a space X_E and endpoint mapping $\phi_E : X_E \to X$, where X_E has a natural uniform structure (but may not be compact!)
- With sufficient connectivity properties, in particular if E is a centourage, X_E is connected and ϕ_E is a regular covering map with deck group $\pi_E(X)$ consisting of all E-homotopy classes of E-loops. We call X_E a centourage cover.
- Note that ε-covers of a geodesic space (compact or not) are centourage covers

The Centourage Spectrum

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Theorem

Let X be a Peano continuum. Then for any entourage F, there are finitely many centourage groups K_E such that $F \subset E$. In particular, if X is semi-locally simply connected then X has finitely

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- On the other hand, the only centourage groups of the circle are the trivial group and $\mathbb{Z}$ (Ellie Abernathy, in preparation)
- Hence this algebraic invariant distinguishes between the circle and the Möbius band even though one is a deformation retraction of the other!

Thoughts on Resistance Metrics

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- Essential circles are "energy minimizing" in a discrete homotopy class and determine a subset of the length spectrum, which, for Riemannian manifolds, is related to the spectrum of the Laplacian.
- But resistance metrics on fractals are not geodesic metrics
- Yet resistance metrics do have metric entourages, and one can ask (1) whether they are centourages, (2) whether there are analogs to essential circles, and if so (3) are those analogs somehow related to the Laplacian on the space?

References

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Thank You