On embeddings of series-parallel graphs in Banach spaces.

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What are series parallel graphs

Definition
A series parallel graph $G = (V, E)$ with terminals $s, t \in V$ is either a single edge $(s, t)$, or a series combination or a parallel combination of two series parallel graphs $G_1$ and $G_2$ with terminals $s_1, t_1$ and $s_2, t_2$. The series combination of $G_1$ and $G_2$ is formed by setting $s = s_1$, $t = t_2$ and identifying $s_2 = t_1$ the parallel combination is formed by identifying $s = s_1 = s_2$, $t = t_1 = t_2$. 

Image from Wikipedia
A generalization of series parallel graphs

Graphs whose every maximal 2-connected component is a series parallel graph, these are graphs of treewidth 2, they are characterized by exclusion of $K_4$ as a minor.

Examples:
- Trees
- Diamonds
- Laakso graphs
Results about embeddings of series parallel graphs

- **Bourgain 1986**
  \( X \) is not super-reflexive \( \iff \sup_n c_X(T_n) < \infty \)

- **Gupta, Newman, Rabinovich, Sinclair 2004**
  \( c_1(\mathcal{F}_{K_4}) \leq 14 \), where \( \mathcal{F}_{K_4} \) is a family of graphs with a forbidden minor \( K_4 \)
  Improved to \( c_1(\mathcal{F}_{K_4}) = 2 \), by Chakrabarti, Jaffe, Lee, Vincent 2008, and Lee, Raghavendra 2010
  Conjecture (GNRS2004) For every family \( \mathcal{F} \) of finite graphs
  \( c_1(\mathcal{F}) < \infty \iff \mathcal{F} \) forbids some minor.

- **Lee, Mendel, Naor 2004**
  For any \( r \in \mathbb{N} \) there exists a constant \( C(r) \) such that for every \( 0 < \alpha < 1 \), an \( \alpha \)-snowflake version of any \( K_r \)-excluded finite graph embeds into \( \ell_2 \) with distortion at most \( C(r)/\sqrt{1 - \alpha} \).
• Chekuri, Gupta, Newman, Rabinovich, Sinclair 2006
  \[ c_1(\text{Outerplanar}(n)) \leq 2^{O(n)}, \quad \forall n \in \mathbb{N} \]

• Lee, Sidiropoulos 2013
  Every minor-closed family of finite graphs \( \mathcal{F} \) which does not contain every possible tree satisfies \( c_1(\mathcal{F}) < \infty \).

• Open Is \( c_1(\text{Planar Graphs}) < \infty \)?

• Johnson, Schechtman 2009
  \( X \) is not super-reflexive \( \iff \sup_n c_X(D_n) < \infty \)
  \( \iff \sup_n c_X(L_n) < \infty \)
  Here \( D_n \) denotes the diamond graphs, and \( L_n \) denotes the Laakso graphs

Remark “It might very well be that Theorem 1 extends to any series parallel graph. We didn’t pursue this farther.”
Diamond graphs

Definition (Gupta, Newman, Rabinovich, Sinclair, 2004)

The *diamond graph* of level 0, denoted $D_0$, contains two vertices joined by an edge. The *diamond graph* $D_n$ is obtained from $D_{n-1}$ by replacing every edge $uv \in E(D_{n-1})$, by a quadrilateral $u, a, v, b$, with edges $ua, av, vb, bu$. 

![Diamond graphs](image-url)
Outline of the talk

1. Examples of series parallel graphs which embed into arbitrary Banach spaces of low dimension,

Problem (Schechtman)

Fix $n \in \mathbb{N}$ satisfying $n \gg 1$ and a constant $K \gg 1$. Characterize all metric spaces admitting embeddings with distortion $\leq K$ into each $n$-dimensional Banach space.

Classical Answers

(A) Metric spaces admitting low-distortion embeddings into log $n$-dimensional Euclidean spaces - consequence of the Dvoretzky theorem

(B) Equilateral spaces of size $\approx a^n$, where $a$ depends on $K$. A metric space is called equilateral if all non-zero distances are equal to each other.

(C) Metric spaces from the classes mentioned in (A) and (B) can be combined using a general construction, called the metric composition. This includes ultrametric spaces.
Question of Gideon Schechtman

Can one suggest examples of metric spaces admitting embeddings with distortion $\leq K$ into each $n$-dimensional Banach space, which are completely different from the examples mentioned above?

Theorem (Ostrovskii, R 2015)
Yes.
Question of Gideon Schechtman

Can one suggest examples of metric spaces admitting embeddings with distortion $\leq K$ into each $n$-dimensional Banach space, which are completely different from the examples mentioned above?

Theorem (Ostrovskii, R 2015)

There exists a family of graphs $\{W_n\}_{n \in \mathbb{N}}$ with $|W_n| \to \infty$, so that for every $C \geq 1$ there exists a constant $D = D(C) \geq 1$ such that for all $n$, the graphs $W_n$ $D$-embed into every Banach space $X$ with a Schauder basis with basis constant smaller than or equal to $C$ and

$$\dim X \geq \frac{1}{2} \left( \log_2 |W_n| \right)^2.$$
Theorem (Szarek and Tomczak-Jaegermann 2009)

There exist absolute constants $A, B, C > 0$ so that for every $n \geq A$ and for every $n$-dimensional normed space $X$, there exists a subspace $Y \subseteq X$ so that $\dim Y \geq B \exp\left(\frac{1}{2} \sqrt{\ln n}\right)$ and $Y$ is $C$-isomorphic to an $\ell_p$-space for some $p \in \{1, 2, \infty\}$.

Corollary

There exist constants $c, N, D > 0$ so that for every $n \geq N$, the graph $W_n$ can be embedded with the distortion not exceeding $D$ in every Banach space $X$ so that

$$\dim X \geq \exp(c(\log \log |W_n|)^2).$$
Diamond graphs

Definition (Gupta, Newman, Rabinovich, Sinclair, 2004)

The *diamond graph* of level 0, denoted $D_0$, contains two vertices joined by an edge.

The *diamond graph* $D_n$ is obtained from $D_{n-1}$ by replacing every edge $uv \in E(D_{n-1})$, by a quadrilateral $u, a, v, b$, with edges $ua, av, vb, bu$. 

![Diagrams of $D_0$, $D_1$, and $D_2$]
Weighted diamond graphs

Let \( \varepsilon \in (0, \frac{1}{2}) \). The sequence \( \{W_n\}_{n=0}^{\infty} \) of weighted diamonds is defined in terms of diamonds \( \{D_n\}_{n=0}^{\infty} \) as follows:

- \( W_0 = D_0 \). The one edge of \( D_0 \) is given weight 1.
- \( W_1 = D_1 \cup W_0 \) with edges of \( D_1 \) given weights \( (\frac{1}{2} + \varepsilon) \); weight of the edge of \( W_0 \) stays as 1.
- \( W_2 = D_2 \cup W_1 \) with edges of \( D_2 \) given weights \( (\frac{1}{2} + \varepsilon)^2 \); weights of the edges of \( W_1 \) stay as they were.
• $W_n = D_n \cup W_{n-1}$ with edges of $D_n$ given weights $(\frac{1}{2} + \varepsilon)^n$; weights of the edges of $W_{n-1}$ stay as they were in the previous step of the construction.

• Graphs $\{W_n\}$ are endowed with the shortest path distance.

Proposition

$W_n$’s are bilipschitz equivalent to snowflakes of $D_n$’s (with distortion $\leq \frac{8}{1-4\varepsilon^2}$).

Definition

Let $(X, d_X)$ be a metric space and $0 < \alpha < 1$. The space $X$ endowed with a modified metric $(d_X(u, v))^\alpha$ is called a \textit{snowflake} of $(X, d_X)$ (or an $\alpha$-\textit{snowflaked version} of $(X, d_X)$).

For $n \geq 2$,

$$\frac{1}{2}4^n \leq |W_n| < 4^n.$$
$W_n$’s do not embed into low dimensional Euclidean spaces

The distortions of embeddings of $W_n$ into $\ell_2^{k(n)}$ can be uniformly bounded only if $k(n) \geq cn = c \log(|W_n|)$ for some $c > 0$.

Thus the Dvoretzky theorem only guarantees that $W_n$ embed in any space of dimension $\geq C_1 |W_n|^{C_2}$, for some $C_1, C_2 > 0$.

Our proof that $W_n$’s admit bounded-distortion embeddings into all Banach spaces with

$$\dim X \geq \frac{1}{2} \left( \log_2 |W_n| \right)^2,$$

if $X$ has a basis, or, in general,

$$\dim X \geq \exp(c(\log \log |W_n|)^2)$$

uses a mixture of $\delta$-net arguments and some “linear” manipulations.
More general examples

Definition
We define inductively a sequence $\{G_n\}_{n=0}^\infty$ of series parallel graphs, which we call *corals*.

Let $\lambda \in (\frac{1}{2}, 1)$ and $\{N_i\}_{i=0}^\infty$ be a sequence of natural numbers so that $N_0 = 2$ and $N_i \geq 1$ for all $i \geq 1$.

Vertices and edges of a coral come in *generations* denoted $\{V_i\}_{i=0}^\infty$ and $\{E_i\}_{i=0}^\infty$, respectively.

- $G_0 = D_0$, i.e. $V_0$ consists of two vertices $v_0, v_1$ joined by one edge of weight 1. Thus $G_0 = (V_0, E_0)$, where $|V_0| = 2$, $|E_0| = 1$. 

Suppose that $\bigcup_{i=0}^{k} V_i$, $\bigcup_{i=0}^{k} E_i$, and $G_k$ have been defined. Let $V_{k+1}$ be a set of cardinality $N_{k+1}$, disjoint with $\bigcup_{i=0}^{k} V_i$. The vertex set of the graph $G_{k+1}$ is $\bigcup_{i=0}^{k+1} V_i$. The set $E_{k+1}$ of new edges is a subset of edges joining the vertices of $V_{k+1}$ with $\bigcup_{i=0}^{k} V_i$. Every edge in $E_{k+1}$ is given weight $\lambda^{k+1}$. Edges in $E_{k+1}$ are chosen so that each vertex in $V_{k+1}$ has degree 1 or 2 and if a vertex $v \in V_{k+1}$ has degree 2 then it is adjacent to vertices $u, w \in \bigcup_{i=0}^{k} V_i$ which are joined by an edge $uw$ in $E_k$, i.e. $uw$ is of length $\lambda^k$ in $G_k$. 
Figure: An example of a coral with a few generations
We define the function $L : \mathbb{N} \to \mathbb{N}$,

$$L(i) = \begin{cases} 
1 & \text{if } i = 1, 2 \\
2 & \text{if } i = 3, 4 \\
\lceil \log_4 i \rceil & \text{if } i \geq 5.
\end{cases}$$

$L(i)$ shows the dimension which is sufficient to accommodate $i$ $\delta$-separated points, for $\delta = 1/16$, in the unit sphere.

**Theorem**

*Let $C \geq 1$ and $\lambda \in (1/2, 1)$. Then there exists a constant $D = D(C, \lambda)$, so that every coral $G_n$ with parameters $\lambda$ and $\{N_i\}_{i=0}^n \subset \mathbb{N}$, $D$-embeds into any Banach space $X$ which contains a basic sequence with basis constant $\leq C$ and of length

$$\sum_{i=0}^{n} L(N_i).$$*
Part 2. Embeddings of multibranching diamonds

Theorem
Let $X$ be a non-superreflexive Banach space. Then for every $n, k \in \mathbb{N}$, and for every $\varepsilon > 0$, the multibranching diamond $D_{n,k}$ (n levels, k branches) embeds into $X$ with distortion $\leq 9 + \varepsilon$. 
Definition (Brunel, Sucheston 1975)
A sequence \( \{ e_n \} \) is called
- equal signs additive (ESA) if for any finitely non-zero sequence \( \{ a_i \} \) of real numbers such that \( \text{sign } a_k = \text{sign } a_{k+1} \),
  \[
  \left\| \sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1}) e_k + \sum_{i=k+2}^{\infty} a_i e_i \right\| = \left\| \sum_{i=1}^{\infty} a_i e_i \right\|. 
  \]
- subadditive (SA) if for any finitely non-zero sequence \( \{ a_i \} \)
  \[
  \left\| \sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1}) e_k + \sum_{i=k+2}^{\infty} a_i e_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i e_i \right\|. 
  \]
- invariant under spreading (IS) if for any finitely non-zero sequence \( \{ a_i \} \) and any increasing \((k_i)_i\)
  \[
  \left\| \sum_{i=1}^{\infty} a_i e_i \right\| = \left\| \sum_{i=1}^{\infty} a_i e_{k_i} \right\|. 
  \]

Theorem (Brunel, Sucheston 1975)  ESA \( \iff \) (SA and IS)
Theorem (Brunel, Sucheston 1975)

For each non-superreflexive space $X$ there is a Banach space $E$ with an ESA basis which is finitely representable in $X$. 
An embedding of $D_{1,k}$ into the space with the ESA basis.

We will work with sequences whose entries are 0 and $\pm 1$. We shall write $+1$ as $+$ and $-1$ as $-$. 

We map the bottom of $D_{1,k}$ to 0, and the top to 

$$ (+ + - - | + + - - | + + - - | + + - - | .... | + + - - | 0000 \ldots ) , $$

the sequence contains $2^k$ blocks of $++--$. 

Note that the element $(++--)$ has 2 metric midpoints 

$(0 + - 0)$ and $(+00-)$ whose distance from each other is 

$$ \|(+-++)\| \geq \|(+-00)\| = \frac{1}{2} \|(++--)\| $$
The element
\[ v_1 = (\ldots | 0000 | \ldots ) \]
has many midpoints.

Let \( r_1, \ldots, r_k \) be the Rademachers on \( \{1, 2, 3, \ldots, 2^k\} \).

For \( 1 \leq i \leq k \), we define an element \( m_i \) using values of \( r_i \):

\[ \nu\text{-th block of } m_i = \begin{cases} 
0 + 0 & \text{if } r_i(\nu) = 1, \\
+00 - & \text{if } r_i(\nu) = -1.
\end{cases} \]

By ESA, \( \forall i \)
\[ \|m_i\| = \frac{1}{2}\|v_1\| \]
and
\[ \|m_i - m_j\| \geq \frac{1}{4}\|v_1\| \]
\[ h^{(n)} = \sum_{l=1}^{2^n} e_l - \sum_{l=2^n+1}^{2^{n+1}} e_l. \]

The element \( h^{(n)} \) is supported on the interval \([1, 2^{n+1}]\). Let

\[ h_+^{(n)} = 0 \ldots 0 + \cdots + - \cdots - 0 \ldots 0, \]

and

\[ h_-^{(n)} = + \cdots + 0 \ldots 0 0 \ldots 0 - \cdots -. \]

Note that by IS and ESA of the basis, we have

\[ \| h_+^{(n)} \| = \| h_-^{(n)} \| = \frac{1}{2} \| h^{(n)} \| = 2^{n-1} \| e_1 - e_2 \|. \]
For any $\alpha = 1, \ldots, n$, and $\{\varepsilon_i\}_{i=1}^{\alpha} \in \{\pm 1\}_{i=1}^{\alpha}$, if $h_{\varepsilon_1, \ldots, \varepsilon_{\alpha-1}}$ is already defined, we define $h_{\varepsilon_1, \ldots, \varepsilon_{\alpha-1}, +}$ to be the element which has $+$ on $2^{n-\alpha}$ largest coordinates at which the element $h_{\varepsilon_1, \ldots, \varepsilon_{\alpha-1}}$ had $+$, and $h_{\varepsilon_1, \ldots, \varepsilon_{\alpha-1}, +}$ has $-$ on $2^{n-\alpha}$ smallest coordinates at which the element $h_{\varepsilon_1, \ldots, \varepsilon_{\alpha-1}}$ had $-$. We define

$$h_{\varepsilon_1, \ldots, \varepsilon_{\alpha-1}, -} \overset{\text{def}}{=} h_{\varepsilon_1, \ldots, \varepsilon_{\alpha-1}, +} - h_{\varepsilon_1, \ldots, \varepsilon_{\alpha-1}, +}.$$

\[
\begin{align*}
h_{++}^{(n)} &= 0 \cdots 0 \underbrace{0 \cdots 0}_{2^{n-1}} + \cdots + - \cdots - 0 \cdots 0 \underbrace{0 \cdots 0}_{2^{n-1}} , \\
h_{+-}^{(n)} &= 0 \cdots 0 + \cdots + 0 \underbrace{0 \cdots 0}_{2^{n-1}} - \cdots - 0 \cdots 0 , \\
h_{-+}^{(n)} &= 0 \cdots 0 + \cdots + 0 \cdots 0 + \cdots + 0 \cdots 0 , \\
h_{--}^{(n)} &= + \cdots + 0 \cdots 0 + \cdots + 0 \cdots 0 ,
\end{align*}
\]

and so on.
We map the bottom vertex of $D_{n,k}$ to be zero, and the top vertex $x_0^{(n)}$ to the sum $2^M$ disjoint shifted copies of the element $h^{(n)}$, where $M$ is big enough,

$$x_0^{(n)} = \sum_{\nu=0}^{2^M-1} S^{2^{n+1} \nu} (h^{(n)}).$$

By IS and ESA of the basis we have

$$\|x_0^{(n)}\| = 2^n \left\| \sum_{\nu=0}^{2^M-1} S^{2^{n+1} \nu} (e_1 - e_2) \right\| = 2^n \left\| \sum_{\nu=0}^{2^M-1} S^{2\nu} (e_1 - e_2) \right\|.$$
Let $r_j, j \in \{1, \ldots, M\}$, be the rademachers on $\{1, \ldots, 2^M\}$. For $\alpha \leq n$, and $(j_1, \ldots, j_\alpha) \in \{1, \ldots, M\}^\alpha$, we denote by $r_{(j_1,\ldots,j_\alpha)}$ the $\alpha$-tuple of the rademachers

$$r_{(j_1,\ldots,j_\alpha)} = (r_j)_{i=1}^\alpha.$$

We define the image of a vertex $v_{\lambda;j_1,\ldots,j_s(\lambda)}^{(n)}$ from level $\lambda$ to be

$$x_{\lambda;j_1,\ldots,j_s(\lambda)}^{(n)} = \sum_{\nu=0}^{2^M-1} S^{2n+1}_\nu \left( \sum_{\alpha=1}^{s(\lambda)} \lambda_\alpha h_{r_{\varphi(j_1,\ldots,j_\alpha)}}^{(n)}(\nu) \right).$$
Thank you.