On embeddings of series-parallel graphs in Banach spaces.

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What are series parallel graphs

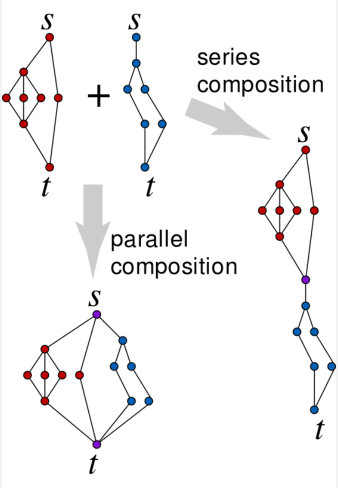


Image from Wikipedia

Definition

A series parallel graph $G = (V, E)$ with terminals $s, t \in V$ is either a single edge (s, t), or a series combination or a parallel combination of two series parallel graphs G_1 and G_2 with terminals s_1, t_1 and s_2, t_2. The series combination of G_1 and G_2 is formed by setting $s = s_1, t = t_2$ and identifying $s_2 = t_1$ the parallel combination is formed by identifying $s = s_1 = s_2, t = t_1 = t_2$.

A generalization of series parallel graphs

Graphs whose every maximal 2-connected component is a series parallel graph, these are graphs of treewidth 2,

they are characterized by exclusion of K_4 as a minor.

Examples:
- Trees
- Diamonds
- Laakso graphs

Results about embeddings of series parallel graphs

- Bourgain 1986
  X is not super-reflexive $\iff \sup_n c_X(T_n) < \infty$
- Gupta, Newman, Rabinovich, Sinclair 2004
  $c_1(\mathcal{F}_{K_4}) \leq 14$, where $\mathcal{F}_{K_4}$ is a family of graphs with a forbiden minor K_4
  Improved to $c_1(\mathcal{F}_{K_4}) = 2$, by Chakrabarti, Jaffe, Lee, Vincent 2008, and Lee, Raghavendra 2010
  Conjecture (GNRS2004) For every family $\mathcal{F}$ of finite graphs $c_1(\mathcal{F}) < \infty \iff \mathcal{F}$ forbids some minor.
- Lee, Mendel, Naor 2004
  For any $r \in \mathbb{N}$ there exists a constant $C(r)$ such that for every $0 < \alpha < 1$, an α-snowflake version of any K_r-excluded finite graph embeds into ℓ_2 with distortion at most $C(r)/\sqrt{1-\alpha}$.

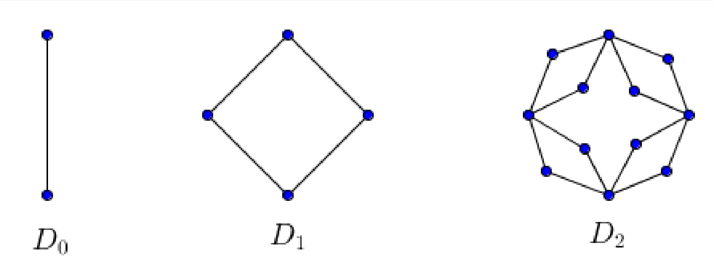
- Chekuri, Gupta, Newman, Rabinovich, Sinclair 2006
  $c_1(\texttt{Outerplanar(n)}) \leq 2^{O(n)}, \quad \forall n \in \mathbb{N}$
- Lee, Sidiropoulos 2013
  Every minor-closed family of finite graphs $\mathcal{F}$ which does not contain every possible tree satisfies $c_1(\mathcal{F}) < \infty$.
- Open Is $c_1(\texttt{Planar Graphs}) < \infty$?
- Johnson, Schechtman 2009
  X is not super-reflexive $\iff \sup_n c_X(D_n) < \infty$
  $\iff \sup_n c_X(L_n) < \infty$
  Here D_n denotes the diamond graphs, and L_n denotes the Laakso graphs

  Remark "It might very well be that Theorem 1 extends to any series parallel graph. We didn't pursue this farther."

Diamond graphs

Definition (Gupta, Newman, Rabinovich, Sinclair, 2004)

The *diamond graph* of level 0, denoted D_0, contains two vertices joined by an edge.
The *diamond graph D_n* is obtained from D_{n-1} by replacing every edge $uv \in E(D_{n-1})$, by a quadrilateral u, a, v, b, with edges ua, av, vb, bu.



Outline of the talk

1. Examples of series parallel graphs which embed into arbitrary Banach spaces of low dimension,

2. A construction of embeddings of arbitrarily large multi-branching diamonds into arbitrary non-superreflexive Banach spaces.

Problem (Schechtman)

Fix $n \in \mathbb{N}$ satisfying $n \gg 1$ and a constant $K \gg 1$. Characterize all metric spaces admitting embeddings with distortion $\leq K$ into each n-dimensional Banach space.

Classical Answers

(A) Metric spaces admitting low-distortion embeddings into $\log n$-dimensional Euclidean spaces - consequence of the Dvoretzky theorem

(B) Equilateral spaces of size $\approx a^n$, where a depends on K.

A metric space is called *equilateral* if all non-zero distances are equal to each other.

(C) Metric spaces from the classes mentioned in **(A)** and **(B)** can be combined using a general construction, called the *metric composition*.

This includes ultrametric spaces.

Question of Gideon Schechtman

Can one suggest examples of metric spaces admitting embeddings with distortion $\leq K$ into each n-dimensional Banach space, which are completely different from the examples mentioned above?

Theorem (Ostrovskii, R 2015)

Yes.

Question of Gideon Schechtman

Can one suggest examples of metric spaces admitting embeddings with distortion $\leq K$ into each n-dimensional Banach space, which are completely different from the examples mentioned above?

Theorem (Ostrovskii, R 2015)

There exists a family of graphs $\{W_n\}_{n\in\mathbb{N}}$ with $|W_n| \to \infty$, so that for every $C \geq 1$ there exists a constant $D = D(C) \geq 1$ such that for all n, the graphs W_n D-embedd into every Banach space X with a Schauder basis with basis constant smaller than or equal to C and

$$\dim X \geq \frac{1}{2}(\log_2 |W_n|)^2.$$

Theorem (Szarek and Tomczak-Jaegermann 2009)

There exist absolute constants $A, B, C > 0$ so that for every $n \geq A$ and for every n-dimensional normed space X, there exists a subspace $Y \subseteq X$ so that $\dim Y \geq B \exp(\frac{1}{2}\sqrt{\ln n})$ and Y is C-isomorphic to an ℓ_p-space for some $p \in \{1, 2, \infty\}$.

Corollary

There exist constants $c, N, D > 0$ so that for every $n \geq N$, the graph W_n can be embedded with the distortion not exceeding D in every Banach space X so that

$$\dim X \geq \exp(c(\log\log|W_n|)^2).$$

Diamond graphs

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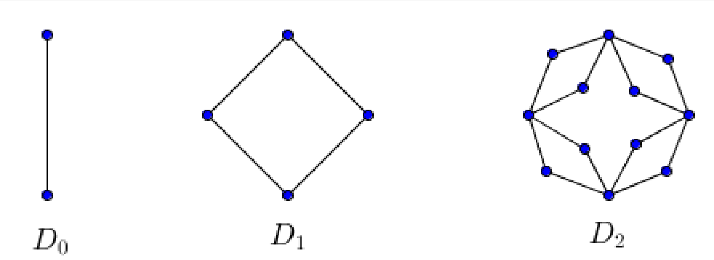
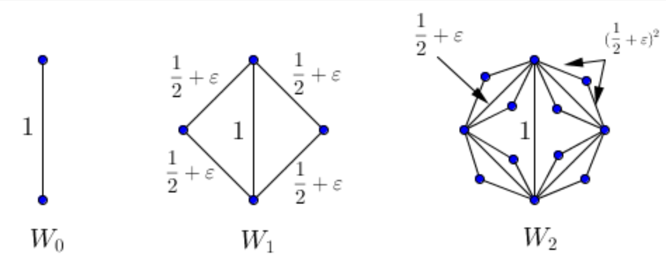


Weighted diamond graphs

Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. The sequence $\{W_n\}_{n=0}^{\infty}$ of *weighted diamonds* is defined in terms of diamonds $\{D_n\}_{n=0}^{\infty}$ as follows:



- $W_0 = D_0$. The one edge of D_0 is given weight 1.
- $W_1 = D_1 \cup W_0$ with edges of D_1 given weights $\left(\frac{1}{2} + \varepsilon\right)$; weight of the edge of W_0 stays as 1.
- $W_2 = D_2 \cup W_1$ with edges of D_2 given weights $\left(\frac{1}{2} + \varepsilon\right)^2$; weights of the edges of W_1 stay as they were.

- $W_n = D_n \cup W_{n-1}$ with edges of D_n given weights $\left(\frac{1}{2}+\varepsilon\right)^n$; weights of the edges of W_{n-1} stay as they were in the previous step of the construction.
- Graphs $\{W_n\}$ are endowed with the shortest path distance.

Proposition

W_n's are bilipschitz equivalent to snowflakes of D_n's (with distortion $\leq \frac{8}{1-4\varepsilon^2}$).

Definition

Let (X, d_X) be a metric space and $0 < \alpha < 1$. The space X endowed with a modified metric $(d_X(u,v))^\alpha$ is called a *snowflake* of (X, d_X) (or an *α-snowflaked version* of (X, d_X)).

For $n \geq 2$,

$$\frac{1}{2}4^n \leq |W_n| < 4^n.$$

W_n's do not embed into low dimensional Euclidean spaces

The distortions of embeddings of W_n into $\ell_2^{k(n)}$ can be uniformly bounded only if $k(n) \geq cn = c\log(|W_n|)$ for some $c > 0$.

Thus the Dvoretzky theorem only guarantees that W_n embed in any space of dimension $\geq C_1|W_n|^{C_2}$, for some $C_1, C_2 > 0$.

Our proof that W_n's admit bounded-distortion embeddings into all Banach spaces with

$$\dim X \geq \frac{1}{2}(\log_2|W_n|)^2,$$

if X has a basis, or, in general,

$$\dim X \geq \exp(c(\log\log|W_n|)^2)$$

uses a mixture of δ-net arguments and some “linear” manipulations.

More general examples

Definition

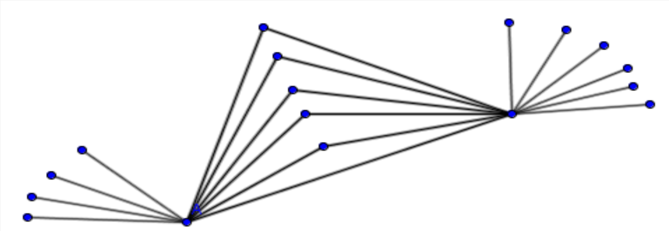
We define inductively a sequence $\{G_n\}_{n=0}^{\infty}$ of series parallel graphs, which we call *corals*.

Let $\lambda \in \left(\frac{1}{2}, 1\right)$ and $\{N_i\}_{i=0}^{\infty}$ be a sequence of natural numbers so that $N_0 = 2$ and $N_i \geq 1$ for all $i \geq 1$.

Vertices and edges of a coral come in *generations* denoted $\{V_i\}_{i=0}^{\infty}$ and $\{E_i\}_{i=0}^{\infty}$, respectively.

- $G_0 = D_0$, i.e. V_0 consists of two vertices v_0, v_1 joined by one edge of weight 1. Thus $G_0 = (V_0, E_0)$, where $|V_0| = 2$, $|E_0| = 1$.

• Suppose that $\bigcup_{i=0}^{k} V_i$, $\bigcup_{i=0}^{k} E_i$, and G_k have been defined. Let V_{k+1} be a set of cardinality N_{k+1}, disjoint with $\bigcup_{i=0}^{k} V_i$. The vertex set of the graph G_{k+1} is $\bigcup_{i=0}^{k+1} V_i$.
The set E_{k+1} of new edges is a subset of edges joining the vertices of V_{k+1} with $\bigcup_{i=0}^{k} V_i$.
Every edge in E_{k+1} is given weight λ^{k+1}.
Edges in E_{k+1} are chosen so that each vertex in V_{k+1} has degree 1 or 2 and if a vertex $v \in V_{k+1}$ has degree 2 then it is adjacent to vertices $u, w \in \bigcup_{i=0}^{k} V_i$ which are joined by an edge uw in E_k, i.e. uw is of length λ^k in G_k.



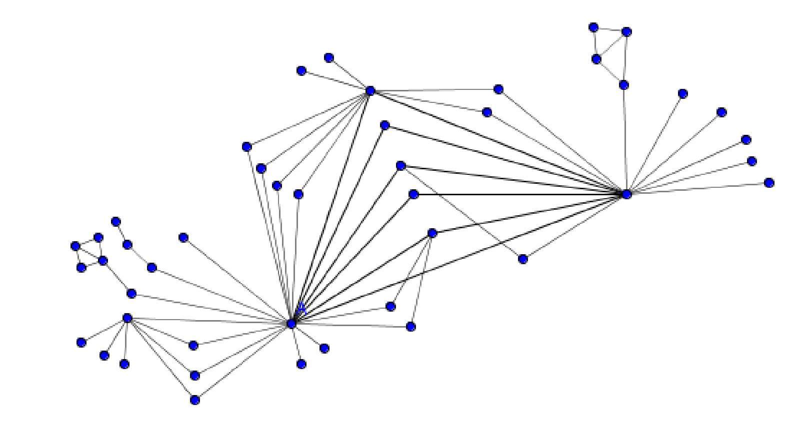


Figure: An example of a coral with a few generations

We define the function $L : \mathbb{N} \to \mathbb{N}$,

$$L(i) = \begin{cases} 1 & \text{if } i = 1, 2 \\ 2 & \text{if } i = 3, 4 \\ \lceil \log_4 i \rceil & \text{if } i \geq 5. \end{cases}$$

$L(i)$ shows the dimension which is sufficient to accommodate i δ-separated points, for $\delta = 1/16$, in the unit sphere.

Theorem

Let $C \geq 1$ and $\lambda \in (1/2, 1)$. Then there exists a constant $D = D(C, \lambda)$, so that every coral G_n with parameters λ and $\{N_i\}_{i=0}^n \subset \mathbb{N}$, D-embeds into any Banach space X which contains a basic sequence with basis constant $\leq C$ and of length

$$\sum_{i=0}^{n} L(N_i).$$

Part 2. Embeddings of multibranching diamonds

Theorem

Let X be a non-superreflexive Banach space.
Then for every $n, k \in \mathbb{N}$, and for every $\varepsilon > 0$, the multibranching diamond $D_{n,k}$ (n levels, k branches) embeds into X with distortion $\leq 9 + \varepsilon$.

Definition (Brunel, Sucheston 1975)
A sequence $\{e_n\}$ is called

- equal signs additive (ESA) if for any finitely non-zero sequence $\{a_i\}$ of real numbers such that $\operatorname{sign} a_k = \operatorname{sign} a_{k+1}$,

$$\Big\| \sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1}) e_k + \sum_{i=k+2}^{\infty} a_i e_i \Big\| = \Big\| \sum_{i=1}^{\infty} a_i e_i \Big\|.$$

- subadditive (SA) if for any finitely non-zero sequence $\{a_i\}$

$$\Big\| \sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1}) e_k + \sum_{i=k+2}^{\infty} a_i e_i \Big\| \leq \Big\| \sum_{i=1}^{\infty} a_i e_i \Big\|.$$

- invariant under spreading (IS) if for any finitely non-zero sequence $\{a_i\}$ and any increasing $(k_i)_i$

$$\Big\| \sum_{i=1}^{\infty} a_i e_i \Big\| = \Big\| \sum_{i=1}^{\infty} a_i e_{k_i} \Big\|.$$

Theorem (Brunel, Sucheston 1975) ESA $\iff$ (SA and IS)

Theorem (Brunel, Sucheston 1975)

For each non-superreflexive space X there is a Banach space E with an ESA basis which is finitely representable in X.

An embedding of $D_{1,k}$ into the space with the ESA basis.

We will work with sequences whose entries are 0 and ± 1. We shall write $+1$ as $+$ and -1 as $-$.

We map the bottom of $D_{1,k}$ to 0, and the top to

$$(++--|++--|++--|++--|....|++--|0000\dots),$$

the sequence contains 2^k blocks of $++--$.

Note that the element $(++--)$ has 2 metric midpoints $(0+-0)$ and $(+00-)$ whose distance from each other is

$$\|(+-+-)\| \geq \|(+-00)\| = \frac{1}{2}\|(++--)\|$$

The element

$$v_1 = (++--|++--|++--|++--|....|++--|0000\ldots),$$

has many midpoints.

Let $r_1, \ldots, r_k$ be the Rademachers on $\{1, 2, 3, \ldots, 2^k\}$.

For $1 \leq i \leq k$, we define an element m_i using values of r_i:

$$\nu\text{-th block of } m_i = \begin{cases} 0+-0 & \text{if } r_i(\nu) = 1, \\ +00- & \text{if } r_i(\nu) = -1. \end{cases}$$

By ESA, $\forall i$

$$\|m_i\| = \frac{1}{2}\|v_1\|$$

and

$$\|m_i - m_j\| \geq \frac{1}{4}\|v_1\|$$

$$h^{(n)} = \underbrace{+\cdots+}_{2^n}\underbrace{-\cdots-}_{2^n} = \sum_{l=1}^{2^n} e_l - \sum_{l=2^n+1}^{2^{n+1}} e_l.$$

The element $h^{(n)}$ is supported on the interval $[1, 2^{n+1}]$. Let

$$h_+^{(n)} = \underbrace{0\ldots0}_{2^{n-1}}\underbrace{+\cdots+}_{2^{n-1}}\underbrace{-\cdots-}_{2^{n-1}}\underbrace{0\ldots0}_{2^{n-1}},$$

and

$$h_-^{(n)} = \underbrace{+\cdots+}_{2^{n-1}}\underbrace{0\ldots0}_{2^{n-1}}\underbrace{0\ldots0}_{2^{n-1}}\underbrace{-\cdots-}_{2^{n-1}}.$$

Note that by IS and ESA of the basis, we have

$$\|h_+^{(n)}\| = \|h_-^{(n)}\| = \frac{1}{2}\|h^{(n)}\| = 2^{n-1}\|e_1 - e_2\|.$$

For any $\alpha = 1, \dots, n$, and $\{\varepsilon_i\}_{i=1}^{\alpha} \in \{\pm 1\}_{i=1}^{\alpha}$, if $h^{(n)}_{\varepsilon_1,\dots,\varepsilon_{\alpha-1}}$ is already defined, we define $h^{(n)}_{\varepsilon_1,\dots,\varepsilon_{\alpha-1},+}$ to be the element which has $+$ on $2^{n-\alpha}$ largest coordinates at which the element $h^{(n)}_{\varepsilon_1,\dots,\varepsilon_{\alpha-1}}$ had $+$, and $h^{(n)}_{\varepsilon_1,\dots,\varepsilon_{\alpha-1},+}$ has $-$ on $2^{n-\alpha}$ smallest coordinates at which the element $h^{(n)}_{\varepsilon_1,\dots,\varepsilon_{\alpha-1}}$ had $-$. We define $h^{(n)}_{\varepsilon_1,\dots,\varepsilon_{\alpha-1},-} \overset{\text{def}}{=} h^{(n)}_{\varepsilon_1,\dots,\varepsilon_{\alpha-1}} - h^{(n)}_{\varepsilon_1,\dots,\varepsilon_{\alpha-1},+}$.

$$h^{(n)}_{++} = \underbrace{0\ldots\ldots\ldots\ldots0}_{2^{n-1}}\underbrace{0\ldots0}_{2^{n-2}}\underbrace{+\cdots+}_{2^{n-2}}\underbrace{-\cdots-}_{2^{n-2}}\underbrace{0\ldots0}_{2^{n-2}}\underbrace{0\ldots\ldots\ldots\ldots0}_{2^{n-1}},$$

$$h^{(n)}_{+-} = \underbrace{0\ldots\ldots\ldots\ldots0}_{2^{n-1}}\underbrace{+\cdots+}_{2^{n-2}}\underbrace{0\ldots0}_{2^{n-2}}\underbrace{0\ldots0}_{2^{n-2}}\underbrace{-\cdots-}_{2^{n-2}}\underbrace{0\ldots\ldots\ldots\ldots0}_{2^{n-1}},$$

$$h^{(n)}_{-+} = \underbrace{0\ldots0}_{2^{n-2}}\underbrace{+\cdots+}_{2^{n-2}}\underbrace{0\ldots\ldots\ldots\ldots0}_{2^{n-1}}\underbrace{0\ldots\ldots\ldots\ldots0}_{2^{n-1}}\underbrace{-\cdots-}_{2^{n-2}}\underbrace{0\ldots0}_{2^{n-2}},$$

$$h^{(n)}_{--} = \underbrace{+\cdots+}_{2^{n-2}}\underbrace{0\ldots0}_{2^{n-2}}\underbrace{0\ldots\ldots\ldots\ldots0}_{2^{n-1}}\underbrace{0\ldots\ldots\ldots\ldots0}_{2^{n-1}}\underbrace{0\ldots0}_{2^{n-2}}\underbrace{-\cdots-}_{2^{n-2}},$$

and so on.

We map the bottom vertex of $D_{n,k}$ to be zero, and the top vertex $x_0^{(n)}$ to the sum 2^M disjoint shifted copies of the element $h^{(n)}$, where M is big enough,

$$x_0^{(n)} = \sum_{\nu=0}^{2^M-1} S^{2^{n+1}\nu}(h^{(n)}).$$

By IS and ESA of the basis we have

$$\|x_0^{(n)}\| = 2^n \Big\| \sum_{\nu=0}^{2^M-1} S^{2^{n+1}\nu}(e_1 - e_2) \Big\| = 2^n \Big\| \sum_{\nu=0}^{2^M-1} S^{2\nu}(e_1 - e_2) \Big\|.$$

Let r_j, $j \in \{1, \dots, M\}$, be the rademachers on $\{1, \dots, 2^M\}$. For $\alpha \leq n$, and $(j_1, \dots, j_\alpha) \in \{1, \dots, M\}^\alpha$, we denote by $r_{(j_1, \dots, j_\alpha)}$ the α-tuple of the rademachers

$$r_{(j_1, \dots, j_\alpha)} = (r_{j_l})_{l=1}^{\alpha}.$$

We define the image of a vertex $v^{(n)}_{\lambda; j_1, \dots, j_{s(\lambda)}}$ from level λ to be

$$x^{(n)}_{\lambda; j_1, \dots, j_{s(\lambda)}} = \sum_{\nu=0}^{2^M - 1} S^{2^{n+1}\nu} \left(\sum_{\alpha=1}^{s(\lambda)} \lambda_\alpha h^{(n)}_{r_{\varphi(j_1, \dots, j_\alpha)}(\nu)} \right).$$

Thank you.