# Bi-Lipschitz embeddings of Grushin spaces

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# The bi-Lipschitz embedding problem

### Definition

A map $f : (X, d_X) \to (Y, d_Y)$ is *bi-Lipschitz* if $C^{-1} d_X(x, y) \leq d_Y(f(x), f(y)) \leq C d_X(x, y)$ for some $C > 1$ and all $x, y \in X$.

### Question

What conditions on a metric space imply that it can or cannot be embedded in a well-known model space (e.g. Euclidean spaces, hyperbolic spaces) under a bi-Lipschitz mapping?

# Historical summary

- Seminal result by Assouad [Ass83] (discussed below)
- Studied intensively in series of papers by Semmes in 1990's (e.g. [Sem93] [Sem96] [Sem99]) using ideas from harmonic analysis and fractal geometry; 'easy conjectures' shown false
- Studied by Lang and Plaut [LP01] using ideas from Alexandrov geometry
- Lafforgue and Naor [LN14], others

# Assoud's Theorem

**Definition**

A metric space $X$ is *doubling* if there is a $C > 1$ such that every ball $B(x, r)$ can be covered by $C$ balls of radius $r/2$.

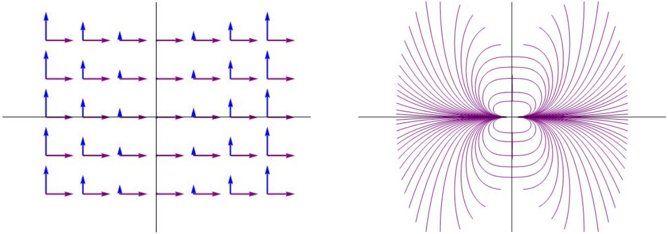
**Theorem ([Ass83])**

Let $(X, d)$ be a doubling metric space. Then, for all $\alpha \in (0, 1)$, the "snowflaked" space $(X, d^\alpha)$ can be bi-Lipschitz embedded in a finite-dimensional Euclidean space.

- Snowflaking distorts geometry of $X$: rectifiablity of curves, Hausdorff dimension.

# The Grushin plane

- Given $\alpha \geq 0$, the vector fields $X = \partial_x$ and $Y = |x|^\alpha \partial_y$ define a sub-Riemannian metric on $\mathbb{R}^2$.
- $X|_{(x,y)}, Y|_{(x,y)}$ form an orthonormal basis for the tangent space at each point in $\mathbb{R}^2 \setminus \{(x,y) : x = 0\}$.
- There are higher-dimensional Grushin spaces; these are defined similarly but with more vector fields.



(a) Vector fields $X$, $Y$ (b) Geodesics from origin

# The Grushin plane

- Length of path $\gamma$ given by

$$\ell(\gamma) = \int_\gamma \sqrt{dx^2 + |x|^{-2\alpha} dx^2}$$

- Distance between two points $z_1, z_2$ is $d_\alpha(z_1, z_2) = \inf \ell(\gamma)$ taken over all absolutely continuous paths from $z_1$ to $z_2$.
- Resulting metric space $(\mathbb{G}^2_\alpha, d_\alpha)$ is Riemannian manifold except on *singular line* $\{(x, y) : x = 0\}$; topologically it is $\mathbb{R}^2$.

# Results on Grushin plane

- Observed by Semmes that Heisenberg group cannot be embedded in Euclidean space, even locally (cf. Cheeger's differentiability theorem) [Sem96]
- Grushin plane studied by Seo [Seo11] in her thesis; shown to be embeddable
- Grushin plane shown by Meyerson [Mey11] to be quasisymmetrically equivalent to Euclidean plane
- Explicit embedding yielding optimal target dimension of 3 constructed by Wu [Wu15]; generalized to $\alpha$-Grusin plane by R. and Vellis [RVar]

# Seo's embeddability criterion

**Theorem ([Seo11])**

A doubling metric space $(X, d)$ admits a bi-Lipschitz embedding into some Euclidean space if and only if the following hold:

(1) There is a closed subset $Y$ of $X$ which admits a bi-Lipschitz embedding into some $\mathbb{R}^{M_1}$.

(2) There is a Christ-Whitney decomposition of $\Omega = X \setminus Y$ such that each cube admits a bi-Lipschitz embedding into some $\mathbb{R}^{M_2}$ with uniform bi-Lipchitz constant.

- Used by Seo to prove embeddability of the Grushin plane

# Christ-Whitney decomposition

**Definition (Christ-Whitney decomposition)**

Let $X$ be a metric space. For an open subset $\Omega \subsetneq X$, a *Christ-Whitney decomposition of* $\Omega$ *with data* $(\delta, c_0, C_1, a)$, where $0 < \delta < 1$, $0 < c_0 < C_1$, $a \geq 4$, is a collection $M_\Omega$ of disjoint open subsets of $X$ satisfying the following properties:

(1) $\bigcup M_\Omega$ is dense in $\Omega$.

(2) For any $Q \in M_\Omega$, there exists $x \in \Omega$ and $k \in \mathbb{Z}$ such that $B(x, c_0\delta^k) \subset Q \subset B(x, C_1\delta^k)$.

(3) $(a-2)C_1\delta^k \leq \operatorname{dist}(Y, Q) \leq \left(\frac{aC_1}{\delta}\right)\delta^k$.

- It is straightforward to verify any proper open subset of a doubling metric space has a Christ-Whitney decomposition, subject to mild restrictions on the data.

# Conformal Grushin spaces

- Define the map $\varphi : \mathbb{G}^2_\alpha \to \mathbb{R}^2$ by

$$(u, v) = \varphi(x, y) = \left( \frac{1}{1+\alpha} |x|^\alpha x, y \right).$$

- This map appears in [Bec01], [MM04], [Mey11]
- The push-forward of the $\alpha$-Grushin line element under $\varphi$ is

$$ds' = \frac{1}{(1+\alpha)^{\alpha/(1+\alpha)} |u|^{\alpha/(1+\alpha)}} \sqrt{du^2 + dv^2}.$$

# Definition of conformal Grushin spaces

## Definition

Let $n \in \mathbb{N}$, let $Y \subset \mathbb{R}^n$ be a nonempty closed set, and let $\beta \in [0, 1)$. The $(Y, \beta)$-*Grushin space* is the space $\mathbb{R}^n$ equipped with the metric determined by the line element

$$ds = \frac{ds_E}{d_E(\cdot, Y)^\beta}.$$

The $(Y, \beta)$-Grushin metric is denoted here by $d_Y$. The Euclidean metric is denoted by $d_E$.

- Similar metric on proper domain $\Omega \subset \mathbb{R}^2$ considered by Gehring–Martio [GM85], Lappalainen [Lap85], Langmeyer [Lan98] (sub-quasihyperbolic metric)
- Take $Y = \{0\} \subset \mathbb{R}^2$ as singular set. The $(Y, \beta)$-Grushin plane is path-isometric to a cone in $\mathbb{R}^3$ with angular defect $2\pi\beta$.

# Main embedding theorem

**Theorem**

Let $n \in \mathbb{N}$, $Y \subset \mathbb{R}^n$ be closed, nonempty, and $\beta \in [0, 1)$. If the $(Y, \beta)$-Grushin space satisfies the Hölder condition $d_Y(x, y) \leq H d_E(x, y)^{1-\beta}$ for some $H > 0$ and all $x, y \in \mathbb{R}^n$, then the $(Y, \beta)$-Grushin space admits a bi-Lipschitz embedding in some Euclidean space of sufficiently high dimension.

- In the following, we will always assume that the $(Y, \beta)$-Grushin space satisfies this Hölder condition.

# Uniform domains and the Hölder condition

**Proposition**

Let $X \subset \mathbb{R}^n$ be a nonempty closed set such that $\Omega = \mathbb{R}^n \setminus X$ is the union of finitely many uniform domains and $\overline{\Omega} = \mathbb{R}^n$. Then for all $\beta \in [0,1)$ and any nonempty closed subset $Y \subset X$, the $(Y, \beta)$-Grushin space satisfies the Hölder condition of the main theorem.

# Main theorem proof outline

- Proof is an application of Seo's embedding criterion. We must check:
  (1) doubling property
  (2) embeddability of singular set $Y$
  (3) uniform embedding of Christ-Whitney cubes
- (2) is immediate by Assouad's theorem since metric on singular set is bi-Lipschitz equivalent to snowflake of Euclidean metric.
- (3) is technical though elementary: we show that $d_Y \simeq d_E(Q, Y)^{-\beta} d_E$ for each cube $Q$ of sufficiently fine Christ-Whitney decomposition.

# Quasisymmetric parametrization

- (1) follows from proving a stronger result that the identity map is a quasisymmetry from $\mathbb{R}^n$ to the $(Y, \beta)$-Grushin space.
- A topological embedding $f : (X, d_X) \to (Y, d_Y)$ is *quasisymmetric* if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta(t)$$

  whenever the distinct points $x, y, z \in X$ satisfy $d_X(x, y) \leq t d_X(x, z)$.
- Quasisymmetric maps preserve the doubling property of a space.

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