Pythagorean powers of hypercubes

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Joint work with

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The linear background

For $1 \le p, q \le 2$, $\ell_p(\ell_q)$ isomorphically embeds into $L_1 = L_1(0,1)$ if and only if $p \le q$.

I am not sure who was the first to prove that for $p > q$ $\ell_p(\ell_q)$ does not embed into L_1 but the best proof of that with the right estimates for the distance of $\ell_p^n(\ell_q^m)$ from a subspace of L_1 follows from an inequality of Kwapień and Schütt.

For all n and all $\{z_{jk}\}_{j,k=1}^n \subseteq L_1$,

$$\frac{1}{n}\sum_{j=1}^n \sum_{\varepsilon\in\{-1,1\}^n} \Big\| \sum_{k=1}^n \varepsilon_k z_{jk} \Big\|_1 \lesssim \frac{1}{n!} \sum_{\pi\in S_n} \sum_{\varepsilon\in\{-1,1\}^n} \Big\| \sum_{j=1}^n \varepsilon_j z_{j\pi(j)} \Big\|_1,$$

where S_n is the group of all permutations of $\{1,\dots,n\}$.

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If $\{z_{jk}\}_{j,k}^{n}$ is the natural basis of $\ell_p^n(\ell_q^n)$,

$$\frac{1}{n}\sum_{j=1}^{n}\Big\|\sum_{k=1}^{n}\varepsilon_k z_{jk}\Big\| = n^{1/q}$$

and

$$\frac{1}{n!}\sum_{\pi\in S_n}\Big\|\sum_{j=1}^{n}\varepsilon_j z_{j\pi(j)}\Big\| = n^{1/p}.$$

So $n^{\frac{1}{p}-\frac{1}{q}} \lesssim d(\ell_p^n(\ell_q^n), SL_1)$.

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The non-linear background

A metric space (X, d_X) is said to admit a bi-Lipschitz embedding into a metric space (Y, d_Y) if there exist $s \in (0, \infty)$, $D \in [1, \infty)$ and a mapping $f : X \to Y$ such that

$$\forall\, x, y \in X, \qquad s d_X(x, y) \leq d_Y(f(x), f(y)) \leq D s d_X(x, y).$$

When this happens we say that that (X, d_X) embeds into (Y, d_Y) with distortion at most D. We denote by $c_Y(X)$ the infimum over such $D \in [1, \infty]$. When $Y = L_p$ we use the shorter notation $c_{L_p}(X) = c_p(X)$.

The non-linear background

We will be interested in lower bounding the distortion of embedding the ℓ_p^n sum of the discrete cube $\mathbb{F}_2^n$ with the ℓ_q norm into L_1. In this lecture we shall restrict ourselves to the case $p = 2$ and $q = 1$.

So we're interested in $c_1(\ell_2^n(\mathbb{F}_2^n))$ where $\mathbb{F}_2^n$ is the n-dimensional discrete hypercube, endowed with the metric inherited from ℓ_1^n via the identification $\mathbb{F}_2^n = \{0,1\}^n \subset \mathbb{R}^n$.

By general principles (ultraproduct, w^*-Gâteaux differentiation), the above stated result of Kwapień and Schütt formally implies that

$$\lim_{n\to\infty} c_1(\ell_2^n(\mathbb{F}_2^n)) = \infty,$$

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Our main result is

Theorem

$c_1(\ell_2^n(\mathbb{F}_2^n)) \asymp \sqrt{n}$.

More generally

Theorem

For all $1 \leq p < q$

$$c_1(\ell_q^n(\mathbb{F}_2^n, \|\cdot\|_p)) \asymp n^{\frac{1}{p}-\frac{1}{q}}.$$

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Recalling

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it is tempting to try and prove the inequality

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\Big|f\Big(x+\sum_{k=1}^{n}e_{jk}\Big)-f(x)\Big|$$

$$\leq \frac{K}{n!}\sum_{\pi\in S_n}\sum_{x\in M_n(\mathbb{F}_2)}\Big|f\Big(x+\sum_{j=1}^{n}e_{j\pi(j)}\Big)-f(x)\Big|$$

for every $n\in\mathbb{N}$ and every $f: M_n(\mathbb{F}_2)\to\mathbb{R}$.
This follows a paradigm set out by Enflo in the 70-s. For $f(x)=\sum_{i=1}^{n}\sum_{k=1}^{n}(-1)^{x_{ik}}z_{ik}$ we recover the linear inequality.

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But the inequality we propose never holds. Even

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)} d_X\Big(f\Big(x+\sum_{k=1}^{n} e_{jk}\Big), f(x)\Big)$$
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for every $n\in\mathbb{N}$ and every $f: M_n(\mathbb{F}_2)\to\mathbb{R}$ does not hold in any metric space X with more than one point:

For n odd and $a\neq b\in X$ define

$$f(x)=a \text{ if } \sum_{i=1}^{n-1}\sum_{k=1}^{n} x_{ik}=0$$

and

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Replace the inequality of Kwapień and Schütt with

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{k=1}^{n}\varepsilon_k z_{jk}\Big\|_1 \le \frac{C}{n^n}\sum_{k\in\{1,\dots,n\}^n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{j=1}^{n}\varepsilon_j z_{jk_j}\Big\|_1$$

for every $n\in\mathbb{N}$ and every $\{z_{jk}\}_{j,k=1}^{n}\subset L_1$.

This, provided it holds, is as good to prove that $c_1(\ell_2^n(\ell_1^n)) \gtrsim \sqrt{n}$.

It turns out that this inequality holds, generalizes to an appropriate metric inequality, and the proof of the metric inequality is even simpler than that of the original KS inequality.

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Theorem

For all $n \in 2\mathbb{N}$ and every $f : M_n(\mathbb{F}_2) \to L_1$ we have

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\left\|f\left(x+\sum_{k=1}^{n}e_{jk}\right)-f(x)\right\|_1$$

$$\leq \frac{2e^2}{e^2-1}\frac{1}{n^n}\sum_{k\in\{1,\dots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\left\|f\left(x+\sum_{j=1}^{n}e_{jk_j}\right)-f(x)\right\|_1.$$

$\frac{2e^2}{e^2-1}$ is the best constant.

The linear inequality follows from the nonlinear one by using

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We'll prove a more general theorem

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Since for $1 \le p \le 2$ "L_p^p isometrically embeds into L_2^2" (i.e., there is a map $g : L_p \to L_2$ with $\|g(x)-g(y)\|_2^2 = \|x-y\|_p^p$ - Schoenberg) and since the statements are purely metric, it is enough to prove the theorem for $p = 2$.

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For all $1 \le p \le 2$*, all* $n \in 2\mathbb{N}$ *and every* $f : M_n(\mathbb{F}_2) \to L_p$ *we have*

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For all $n \in 2\mathbb{N}$ and every $f : M_n(\mathbb{F}_2) \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\left|f\left(x+\sum_{k=1}^{n}e_{jk}\right)-f(x)\right|^2$$
$$\le \frac{2e^2}{e^2-1}\frac{1}{n^n}\sum_{k\in\{1,\dots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\left|f\left(x+\sum_{j=1}^{n}e_{jk_j}\right)-f(x)\right|^2.$$

We can now use simple Fourier Analysis.

The proof

Need to prove.

Theorem

For all $n \in 2\mathbb{N}$ and every $f : M_n(\mathbb{F}_2) \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\left|f\left(x+\sum_{k=1}^{n}e_{jk}\right)-f(x)\right|^2$$
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We can now use simple Fourier Analysis.

The proof

$$f(x) = \sum_{A_1,\ldots,A_n \subset \{1,\ldots,n\}} \widehat{f}(A_1,\ldots,A_n)(-1)^{\sum_{j=1}^n \sum_{k\in A_j} x_{jk}},$$

where for every $A_1,\ldots,A_n \subset \{1,\ldots,n\}$,

$$\widehat{f}(A_1,\ldots,A_n) = \frac{1}{2^{n^2}} \sum_{x\in M_n(\mathbb{F}_2)} (-1)^{\sum_{j=1}^n \sum_{k\in A_j} x_{jk}} f(x).$$

The proof

Then, for every $x \in M_n(\mathbb{F}_2)$ and $j \in \{1, \ldots, n\}$ we have

$$f\Big(x + \sum_{k=1}^{n} e_{jk}\Big) - f(x)$$

$$= \sum_{A_1,\ldots,A_n \subset \{1,\ldots,n\}} \widehat{f}(A_1, \ldots, A_n) \left((-1)^{|A_j|} - 1\right) (-1)^{\sum_{s=1}^{n} \sum_{k \in A_s} x_{sk}}$$

$$= -2 \sum_{\substack{A_1,\ldots,A_n \subset \{1,\ldots,n\} \\ |A_j| \ odd}} \widehat{f}(A_1, \ldots, A_n)(-1)^{\sum_{s=1}^{n} \sum_{k \in A_s} x_{sk}}.$$

And

$$\frac{1}{2^{n^2}} \sum_{j=1}^{n} \sum_{x \in M_n(\mathbb{F}_2)} \left(f\Big(x + \sum_{k=1}^{n} e_{jk}\Big) - f(x) \right)^2$$

$$= 4 \sum_{A_1,\ldots,A_n \subset \{1,\ldots,n\}} \Big| \{ j \in \{1, \ldots, n\} : \ |A_j| \ odd \} \Big| \widehat{f}(A_1, \ldots, A_n)^2.$$

The proof

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$$f\left(x+\sum_{j=1}^{n} e_{jk_j}\right)-f(x)$$

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And

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$$=\sum_{k\in\{1,\dots,n\}^n}\sum_{A_1,\dots,A_n\subset\{1,\dots,n\}}\widehat{f}(A_1,\dots,A_n)^2\left((-1)^{\sum_{j=1}^{n}\mathbf{1}_{A_j}(k_j)}-1\right)^2$$

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So need to prove

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for all $A_1,\dots,A_n\subset\{1,\dots,n\}$.
Note that what this amounts to is just proving the inequality for f being one of the Walsh functions.

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The proof

Put $S = \{j \in \{1,\dots,n\} :\ |A_j|\ odd\}$.

$$\sum_{k\in\{1,\dots,n\}^n}\left(1-(-1)^{\sum_{j=1}^n \mathbf{1}_{A_j}(k_j)}\right) = n^n - \prod_{j=1}^n\sum_{k=1}^n(-1)^{\mathbf{1}_{A_j}(k)}$$

$$= n^n - \prod_{j=1}^n (n-2|A_j|) \ge n^n - \prod_{j=1}^n \big|2|A_j|-n\big| \ge n^n - n^{n-|S|}(n-2)^{|S|}.$$

Since the mapping $|S| \mapsto \left(n^n - n^{n-|S|}(n-2)^{|S|}\right)/|S|$ is decreasing in $|S|$,

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