Distortion of dimension by metric space-valued Sobolev mappings Lecture I

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Outline

Lecture I. Sobolev and quasiconformal mappings in Euclidean space

Lecture II. Sobolev mappings between metric spaces

Lecture III. Dimension distortion theorems for Sobolev and quasiconformal mappings defined from the sub-Riemannian Heisenberg group

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Sobolev functions and mappings

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Question: To what extent can a continuous map $f \in W^{1,p}(\Omega : \mathbb{R}^N)$ distort the dimension of

- a fixed subset $E \subset \Omega$?
- a typical element in a parameterized family $\{E_\lambda\}_{\lambda \in \Lambda}$ of subsets of Ω?

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Quasiconformal mappings are an important motivating class of examples. These are $W^{1,n}_{loc}$ homeomorphisms between domains of $\mathbb{R}^n$ satisfying a uniform relative metric distortion estimate (**dilatation**). The precise relationship between dilatation, Sobolev regularity, and dimension distortion is still not completely understood in arbitrary dimensions.

Some preliminary bounds

1. Every Sobolev mapping has an ACL representative.

[$g : \Omega \to \mathbb{R}$ is ACL if, for each $Q \Subset \Omega$, $g|_\gamma : \gamma \to \mathbb{R}$ is absolutely continuous for $\mathcal{L}^{n-1}$-a.e. line segment γ contained in Q and parallel to any coordinate axis.]

If f is ACL, then for $\mathcal{L}^{n-1}$-a.e. γ, $f(\gamma)$ is rectifiable and hence

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2. According to the **Morrey–Sobolev embedding theorem**, mappings in $W^{1,p}$ with $p > n$ have α-Hölder continuous representatives with $\alpha = 1 - \frac{n}{p}$.

α-Hölder continuity of f implies $\dim f(E) \le \frac{\dim E}{\alpha}$ for any $E \subset \Omega$.
Thus if $f \in W^{1,p}(\mathbb{R}^n : \mathbb{R}^N)$ with $p > n$, then

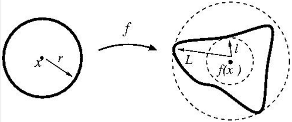
$$\dim f(\gamma) \le \frac{p}{p-n}.$$

Quasiconformal mappings

Definition

A homeo $f : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$, $n \geq 2$, is **(metrically)** H-**quasiconformal** if

$$\limsup_{r \to 0} \frac{\max\{|f(x) - f(y)| : |x - y| = r\}}{\min\{|f(x) - f(z)| : |x - z| = r\}} \leq H \quad \forall\, x$$



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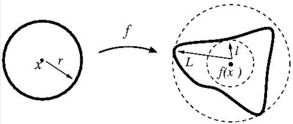

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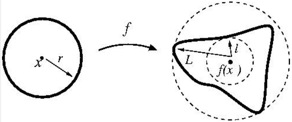

1. Equation (1) imposes a uniform bound on infinitesimal relative metric distortion.

2. Not easy to develop a theory based on this definition, e.g., not clear that compositions, inverses, limits of QC maps are QC

3. 1-QC maps in $\mathbb{R}^2$ are conformal, 1-QC maps in $\mathbb{R}^n$, $n \geq 3$ are Möbius

Quasiconformal mappings: a brief history

- Grötzsch (1928): extremal problems in complex analysis
- 1930–1960: planar theory

  Teichmüller theory/Riemann surfaces/quadratic differentials

  univalent function theory
- 1960–1980s: n dimensional and Riemannian theory

  Mostow rigidity theorem for hyperbolic manifolds

  Kleinian groups, complex dynamics

  differential geometry (Donaldson–Sullivan theory of QC 4-manifolds)
- 1980s–present: QC mappings in non-Riemannian metric spaces

  geometric group theory

  sub-Riemannian geometry

  first-order regularity theory for mappings between metric spaces

Quasiconformal mappings: equivalence of definitions

QC maps have been defined as homeomorphisms satisfying an infinitesimal metric distortion condition. A basic feature of QC mapping theory is that such a condition suffices to derive improved analytic regularity as well as global metric distortion estimates.

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Definition

A homeo $f : X \to Y$ between metric spaces is **quasisymmetric** if there exists a homeo $\eta : [0, \infty) \to [0, \infty)$ s.t.

$$\frac{d(f(a), f(b))}{d(f(a), f(c))} \leq \eta\left(\frac{d(a, b)}{d(a, c)}\right) \quad \forall\, a, b, c \in X$$

Global, easy to see that inverses, compositions, limits (w/ some normalization) of QS mappings are QS.

η-QS maps are H-QC with $H = \eta(1)$

Quasiconformal mappings: equivalence of definitions

The starting point for the analytic theory of quasiconformal mappings is the following regularity theorem of Fred Gehring.

Theorem (F. W. Gehring, 1962)

QC mappings are ACL. If f is QC between domains in $\mathbb{R}^n$, then

$$||Df||^n \leq K \det Df \qquad a.e.$$

where $K = K(H, n)$. In particular, $f \in W^{1,n}_{loc}$.

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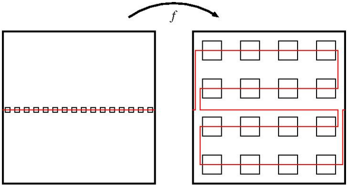
$$\textit{metric QC} \overset{\textit{Gehring}}{\Rightarrow} \textit{analytic QC} \Rightarrow \textit{geometric QC} \overset{\textit{modulus estimates}}{\Rightarrow} \textit{(local) QS}$$

Dimension distortion by quasiconformal maps



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closed disjoint subcubes in Q

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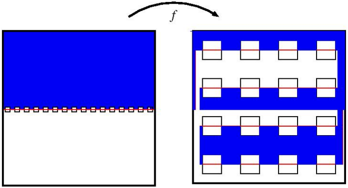

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$f_m = f_{m-1}|_{(\cup Q_w)^c} + \sum_w (\phi'_w)^{-1} \circ f_{m-1} \circ \phi_w|_{Q_w}$

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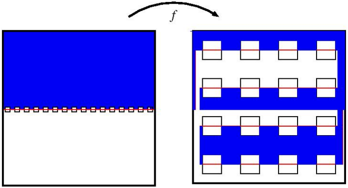

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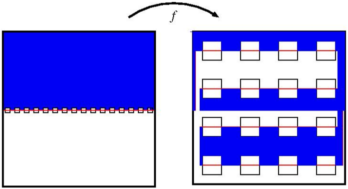

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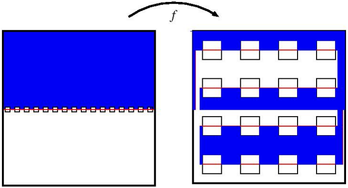

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Theorem (Gehring–Väisälä, 1973)

K-QC maps preserve sets of dimension 0 and n quantitatively. More precisely, let $f : \Omega \to \Omega'$ be K-QC in $\mathbb{R}^n$, $n \geq 2$ and $E \subset \Omega$ with $\dim E = s \in (0, n)$. Then

$$0 < \alpha(n, K, s) \leq \dim f(E) \leq \beta(n, K, s) < n.$$

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Theorem (F. W. Gehring, 1973)

For $n \geq 2$ and $K \geq 1$ $\exists\, p > n$ s.t. f K-QC between domains in $\mathbb{R}^n \Rightarrow f \in W^{1,p}_{loc}$.

Let $p(n, K)$ be the supremum of such $p > n$. Gehring and Väisälä's dimension estimates can be stated in the following symmetric form:

$$\left(1 - \frac{n}{p(n,K)}\right)\left(\frac{1}{\dim E} - \frac{1}{n}\right) \leq \frac{1}{\dim f(E)} - \frac{1}{n} \leq \left(1 - \frac{n}{p(n,K)}\right)^{-1}\left(\frac{1}{\dim E} - \frac{1}{n}\right)$$

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Conjecture: $p(n, K) = \frac{nK}{K-1}$.

Suggested by the **radial stretch map** $f(x) = |x|^{(1/K)-1}x$. Proved by Astala (1994) for $n = 2$.

Dimension distortion by QC and Sobolev mappings

Gehring and Väisälä's proof of their dimension estimates for QC mappings uses the metric distortion condition. However, the upper estimate which they obtain,

$$\dim f(E) \le \beta(n, K, \dim E),$$

can be derived solely from the uniform continuity estimates coming from the improved Sobolev regularity of such mappings.

The lower bound

$$\dim f(E) \ge \alpha(n, K, \dim E)$$

follows by symmetry, applying the previous result to $g = f^{-1}$ and using the fact that the inverse of a K-QC map is also K-QC.

We next indicate how such estimates are derived.

Let f be a continuous mapping in $W^{1,p}(\Omega : \mathbb{R}^N)$, $p > n$.

A telescoping argument and repeated application of the Poincaré inequality $\fint_{Q'} |f - f_{Q'}| \leq C(\operatorname{diam} Q')(\fint_{Q'} |\nabla f|^p)^{1/p}$, yields the **Morrey–Sobolev estimate**

$$|f(x) - f(y)| \leq C(n,p)|x-y|^{1-n/p} \left(\int_Q |\nabla f|^p \right)^{1/p} \quad \forall\, x, y \in Q \Subset \Omega$$

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Let $E \subset \Omega$, $\mathcal{H}^s(E) < \infty$, $s < n$. Choose essentially disjoint dyadic cubes $\{Q_i\}$ s.t.

- $E \subset \cup_i Q_i$,
- $r_i := \operatorname{diam} Q_i < \delta$ (for given $\delta > 0$),
- $\cup_i Q_i \subset U$ (for given U open with $E \subset U \subset \Omega$), and
- $\sum_i r_i^s \le C < \infty$.

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$$\mathcal{H}^\alpha_{\epsilon(\delta,n,p)}(f(E)) \leq \sum_i (\operatorname{diam} f(Q_i))^\alpha \lesssim \sum_i (\operatorname{diam} Q_i)^{(1-n/p)\alpha} \Big(\int_{Q_i} |\nabla f|^p \Big)^{\alpha/p}$$

$$\lesssim \left(\sum_i r_i^{\frac{p-n}{p}\alpha \frac{p}{p-\alpha}} \right)^{1-\frac{\alpha}{p}} \left(\int_{\cup_i Q_i} |\nabla f|^p \right)^{\frac{\alpha}{p}} \leq \left(\sum_i r_i^s \right)^{1-\frac{\alpha}{p}} ||\nabla f||^\alpha_{p,U} \to 0$$

We proved the following: if $f \in W^{1,p}(\Omega : \mathbb{R}^N)$, $p > n$, is continuous and $E \subset \Omega$, then

$$\mathcal{H}^s(E) < \infty,\ s < n \qquad \Rightarrow \qquad \mathcal{H}^\alpha(f(E)) = 0, \qquad \alpha = \frac{ps}{p - n + s}.$$

In particular,

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Remarks: (1) $\alpha \leq \frac{ps}{p-n+s} \Leftrightarrow (1 - \frac{n}{p})(\frac{1}{s} - \frac{1}{n}) \leq \frac{1}{\alpha} - \frac{1}{n}$.

(2) Kaufman (2000) gave a nonconstructive argument showing that the above estimate is sharp in the following sense:

For **any** $s < n < p$ and any closed set $E \subset \mathbb{R}^n$ s.t. $\mathcal{H}^s(E) > 0$, there exists $f \in W^{1,p}(\mathbb{R}^n : \mathbb{R}^n)$ s.t. $\dim f(E) \geq \frac{ps}{p-n+s}$.

Almost sure Sobolev dimension distortion estimates

Fix a line L in $\mathbb{R}^n$ and consider the foliation $\{L + a : a \in L^\perp\}$ of $\mathbb{R}^n$ by parallel affine lines. Recall:

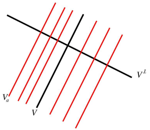
$$\dim f(L+a) \leq 1 \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } a \in L^\perp \text{ and for any } f \in W^{1,p}$$

and

$$\dim f(L+a) \leq \frac{p}{p-n+1} \qquad \text{for all } a \in L^\perp \text{ and for } f \in W^{1,p},\ p > n.$$

Our first main result interpolates between these two statements, using the family of Hausdorff measures $\mathcal{H}^\beta$, $0 \leq \beta \leq n-1$, on the orthogonal subspace $L^\perp$.

Fix an m-dimensional subspace V in $\mathbb{R}^n$.



$$\mathbb{R}^n = \bigcup V_a \qquad V_a = V + a, \quad a \in V^\perp$$

Theorem A (Balogh–Monti–T, 2013)

Let $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$, $p > n$. Fix $m < \alpha \leq \frac{pm}{p-n+m}$. Then

$$\dim f(V_a) \leq \alpha \quad \textit{for } \mathcal{H}^\beta\textit{-a.e. } a$$

where $\beta = \beta(\alpha) = (n-m) - p(1 - \frac{m}{\alpha})$. In fact, $\mathcal{H}^\alpha(f(V_a)) = 0$ for $\mathcal{H}^\beta$-a.e. a.

Note that $\beta(\frac{pm}{p-n+m}) = 0$, so we recover the universal dimension estimate in the borderline case.

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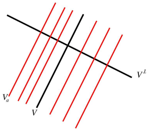

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Remark: The theorem is sharp. For all relevant choices of the data, there exists a $W^{1,p}$ mapping which simultaneously raises the dimension of each member of a β-dimensional family of parallel subspaces V_a to the optimal value α. We will discuss this and related examples soon.

Sobolev mappings at the critical exponent

We always consider the (Hölder) continuous representative of a supercritical $W^{1,p}$ map ($p > n$). Note that $W^{1,n}$ maps in $\mathbb{R}^n$ need not be continuous, or even L^∞_{loc}.

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Theorem B (Hajłasz–T, 2008)

Fix $2 \leq n \leq N < \infty$. *Then there exists* $f \in W^{1,n}([0,1]^n : \mathbb{R}^N)$ *continuous s.t.* $f([0,1]^n) = [0,1]^N$.

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(2) $[0,1]^N$ can be replaced by the Hilbert cube.

(3) False when $n = 1$: every $W^{1,1}$ map on $[0,1]$ is absolutely continuous.

(4) Theorem B guarantees the existence of **Sobolev Peano cubes** in the critical exponent $p = n$. We'll discuss a general theorem of Hahn–Mazurkiewicz type for metric space-valued Sobolev maps in the next lecture.

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Claim

$\int_{E_\alpha} \mathcal{H}^\alpha(f(V_a))\, d\mu(a) = 0.$

If true, this contradicts the definition of the exceptional set E_α and finishes the proof.

Proof of Theorem A (continued)

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- Let $\{R_i\}$ be essentially disjoint dyadic cubes in $V^\perp$ covering E with $\cup_i R_i \subset N_\delta(E)$ and $\sum_i r_i^\beta < \infty$.

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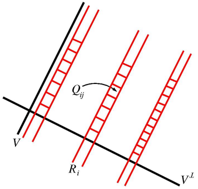

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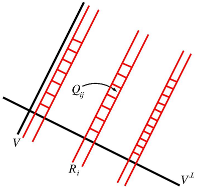

$$\int_{V^\perp}^{*} \mathcal{H}^\alpha_\epsilon(f(V_a))\, d\mu(a) \le \int_{V^\perp}^{*} \sum_{i:a\in R_i} \sum_{j=1}^{N_i} \operatorname{diam} f(Q_{ij})^\alpha \, d\mu(a)$$

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Use the Frostman estimate, Morrey–Sobolev estimate, Hölder (twice).

$$\le C\left(\sum_i \left(r_i^{\beta} r_i^{(1-\frac{n}{p})\alpha} r_i^{-m(1-\frac{\alpha}{p})}\right)^{\frac{p}{p-\alpha}}\right)^{\frac{p-\alpha}{p}} \left(\int_{U_\delta} |Df|^p\right)^{\frac{\alpha}{p}}$$

$$= C\left(\sum_i r_i^{\beta}\right)^{\frac{p-\alpha}{p}} ||Df||^\alpha_{L^p(U_\delta)} \xrightarrow{\delta\to 0} 0.$$

Applications to quasiconformal maps

QC maps always lie in a supercritical Sobolev space $W^{1,p}$ by Gehring's theorem. Since

$$\beta(\alpha) = (n-m) - p\left(1 - \tfrac{m}{\alpha}\right) < (n-m) - n\left(1 - \tfrac{m}{\alpha}\right) = m\left(\frac{n}{\alpha} - 1\right)$$

we get the following conclusion.

Corollary

Let f be QC in $\mathbb{R}^n$, V an m-dimensional subspace, $m < \alpha < n$. Then

$$\mathcal{H}^\alpha(f(V_a)) = 0 \qquad \textit{for } \mathcal{H}^{m(\frac{n}{\alpha}-1)}\textit{-a.e. } a \in V^\perp.$$

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Note that the dilatation K doesn't appear in the corollary. Since the precise value of $p(n,K)$ is still not known when $n \geq 3$, any such estimate is not likely to be sharp in any case. Even in dimension 2, it is challenging to exhibit examples of QC maps realizing sharpness of dimension bounds for large families of parallel lines (cf. work of Smirnov).

Examples

1. (Balogh–Monti–T) For all sufficiently regular sets $E \subset V^\perp$ of dimension β, there exists a $W^{1,p}$ map f s.t. $\dim f(V_a) \geq \alpha$ for $\mathcal{H}^\beta$-a.e. $a \in E$.

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Question: Does there exist a QC map which increases the **Hausdorff** dimension of an uncountable family of parallel lines?

3. (Bishop–Hakobyan–Williams, 2015) Fix $1 < \alpha < 2$. For each $\epsilon > 0$ there exists f QC in $\mathbb{R}^2$ s.t. $\dim f(L_a) \geq \alpha$ for a set of $a \in L^\perp$ of Hausdorff dimension $> (\frac{2}{\alpha} - 1) - \epsilon$. Moreover, $\dim f(F + a) \geq \alpha \cdot \dim F$ for **all** Borel subsets $F \subset L$.

Examples

4. (Balogh–T–Wildrick, 2016) Fix $1 < \alpha < n$. For all $p > n$ and for all $\epsilon > 0$ there exists f in $\mathbb{R}^n$ QC **and in** $W^{1,p}_{loc}$ s.t. $\dim f(L_a) \geq \alpha$ for a set of $a \in L^\perp$ of Hausdorff dimension $> (n-1) - p(1 - \frac{1}{\alpha}) - \epsilon$.

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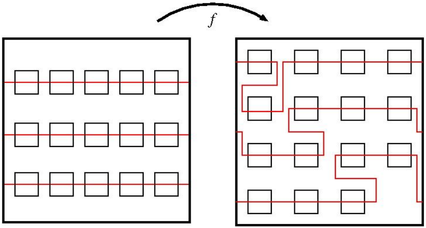

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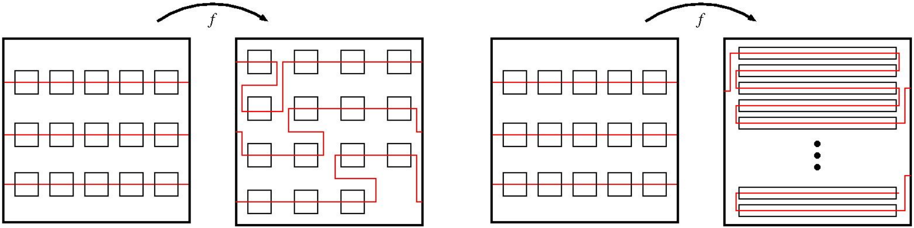

In Lecture II we will introduce the class of Sobolev mappings taking values in a metric space:

$$W^{1,p}(\Omega : Y)$$

and the class of **Newtonian–Sobolev mappings** from a metric measure space (X, d, μ) to a metric space Y:

$$N^{1,p}(X : Y).$$

We will discuss generalizations of the preceding dimension distortion theorems to those settings.

Such results apply to several geometrically natural foliations of the sub-Riemannian Heisenberg group $\mathbb{H}^n$, but specifically do not apply to **left coset horizontal foliations** of $\mathbb{H}^n$. The latter foliations arise in the study of Sobolev and QC mappings in connection with the ACL property.

In Lecture III we will discuss dimension distortion estimates specific to the sub-Riemannian setting which apply specifically in the setting of left coset foliations.