Distortion of dimension by metric space-valued Sobolev mappings Lecture II

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Metric Spaces: Analysis, Embeddings into Banach Spaces, Applications
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Outline

Lecture I. Sobolev and quasiconformal mappings in Euclidean space

Lecture II. Sobolev mappings between metric spaces

Lecture III. Dimension distortion theorems for Sobolev and quasiconformal mappings defined from the sub-Riemannian Heisenberg group

Motivation: quasiconformal mappings in metric spaces

The theory of analysis in metric measure spaces originates in two papers of Juha Heinonen and Pekka Koskela:

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Sobolev mappings into metric spaces: a brief history

Ambrosio (1990): metric space-valued BV mappings

Korevaar–Schoen (1993): $W^{1,2}$ mappings into metric spaces with Alexandrov curvature bounds

Reshetnyak (1997)

T (1998)

Heinonen–Koskela–Shanmugalingam–T (2001, 2015): theory of the Newtonian–Sobolev space $N^{1,p}(X : Y)$ with applications to QS maps

A. and J. Björn, *Nonlinear potential theory on metric spaces*, European Math. Soc., 2011

Sobolev mappings into Banach spaces

Let $\Omega \subset \mathbb{R}^n$ and V a Banach space.

$W^{1,p}(\Omega : V)$ is the space of $f : \Omega \to V$ s.t.

(i) $f \in L^p(\Omega : V)$

(ii) f has a weak gradient $\nabla f \in L^p(\Omega : V^n)$
(integrals against C_0^∞ test functions understood in the Bochner sense).

$W^{1,p}(\Omega : V)$ is a Banach space when equipped with the norm

$$||f||_{1,p} := \left(\int_\Omega ||f||^p \right)^{1/p} + \left(\int_\Omega ||\nabla f||^p \right)^{1/p} .$$

Sobolev mappings into Banach spaces

An alternate approach is to consider post-composition by linear functionals (or more general Lipschitz functions). Reshetnyak (1997) used a similar approach to study metric space-valued Sobolev maps.

The **Reshetnyak–Sobolev space** $R^{1,p}(\Omega : V)$ is the set of $f \in L^p(\Omega : V)$ s.t.

(i) for each $v^* \in V^*$, $\langle v^*, f \rangle \in W^{1,p}(\Omega)$, and

(ii) there exists $0 \leq g \in L^p(\Omega)$ s.t. $|\nabla \langle v^*, f \rangle| \leq g$ a.e., for each $v^* \in V^*$ with $||v^*|| \leq 1$.

$R^{1,p}(\Omega : V)$ is a Banach space when equipped with the norm

$$||f||_{R^{1,p}} := ||f||_p + \inf_g ||g||_p$$

where the infimum is over all g as in (ii).

Proposition

If V is dual to a separable Banach space, then $W^{1,p}(\Omega : V) = R^{1,p}(\Omega : V)$ and $|| \cdot ||_{1,p} \approx || \cdot ||_{R^{1,p}}$.

Sobolev mappings into metric spaces

Y a metric space, $\kappa : Y \to V$ an isometric embedding into a Banach space V.

$W^{1,p}(\Omega : Y)$ is the set of all $f \in W^{1,p}(\Omega : V)$ s.t. $f(\Omega) \subset \kappa(Y)$. We similarly define $R^{1,p}(\Omega : Y)$.

If V is a separable dual, then these two spaces are equal (e.g., for every separable metric space).

Remarks: (1) Sobolev maps between Riem mflds are often defined similarly, using a Nash embedding of the target. Our approach is slightly different, since the notion of 'isometric embedding' in the Nash theorem is intrinsic.

(2) The space $W^{1,p}(\Omega : Y)$ is independent of the choice of κ. However, the induced metric on $W^{1,p}(\Omega : Y)$ **does** depend on κ (cf. work of Hajłasz).

(3) The dimension distortion theorems from Lecture I continue to hold for supercritical Sobolev maps with metric space target.

Sobolev Peano cubes

The classical Hahn–Mazurkiewicz theorem states that a metric space Y is the continuous image of $[0,1]$ iff Y is compact, connected and locally connected.

To generalize this to the Sobolev category we need to impose stronger assumptions on Y.

Since Sobolev maps are ACL, if $Y = f([0,1]^n)$ for some Sobolev map f, then Y must contain many rectifiable curves.

For a rectifiably connected metric space (Y, d_Y), let d_ℓ be the associated **length metric**. Y is **length compact** if (Y, d_ℓ) is compact.

Theorem (Hajłasz–T, 2008)

Let Y be length compact and $n \geq 2$. Then there exists a continuous surjection $f \in W^{1,n}([0,1]^n : Y)$. Moreover, f can be chosen to be locally Lipschitz continuous in the complement of a closed set of Hausdorff dimension zero.

Sobolev Peano cubes: proof sketch

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Key fact: The Sobolev n-capacity of a point $x_0 \in \mathbb{R}^n$ is equal to zero if $n \geq 2$.

(For any $\epsilon > 0$ there is a $W^{1,n}$ function u supported in $B(x_0, \epsilon)$ which constant in a smaller nbhd and s.t. $||\nabla u||_n < \epsilon$.)

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Length compactness $\Rightarrow \exists$ finite 2^{-m}-nets Y_m for each $m \in \mathbb{N}$.

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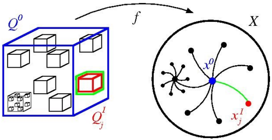
Length compactness $\Rightarrow \exists$ finite 2^{-m}-nets Y_m for each $m \in \mathbb{N}$.

Exhaust Y by a locally finite tree comprised of almost geodesic arcs γ_w^m joining each $y_w^m \in Y_m$ to its parent in Y_{m-1}.



$\{Q_w^m\}$ a combinatorially equiv system of cubes in Q^0.

Define $f : [0,1]^n \to Y$ as follows: $f = y_w^m$ on ∂Q_w^m

$f = \gamma_w^m \circ u_w^m$ interpolates from y_w^m to its parent along γ_w^m in an annulus $\lambda Q_w^m \setminus Q_w^m$.

Upper gradients

(X, d, μ) a metric measure space (μ Borel, $0 < \mu(B) < \infty$ for all balls B.)

Definition (Heinonen–Koskela)

A Borel function $g : X \to [0, \infty]$ is an **upper gradient** of $f : X \to Y$ if

$$d_Y(f(x), f(y)) \leq \int_\gamma g\, ds$$

whenever γ is a rectifiable curve in X joining x to y.

e.g., $|\nabla f|$ is an upper gradient of $f : \Omega \to \mathbb{R}$ for any $f \in C^1(\Omega)$.

Newtonian–Sobolev spaces

Definition (Shanmugalingam)

The **Newtonian–Sobolev space** $N^{1,p}(X)$ consists of all $f \in L^p(X)$ which admit an upper gradient $g \in L^p(X)$.

Facts: (1) $N^{1,p}(X)$ is a Banach space when equipped with the norm $||f||_{1,p} = ||f||_p + \inf_g ||g||_p$.

(2) $N^{1,p}(\Omega) = W^{1,p}(\Omega)$ for Euclidean domains Ω (modulo a caveat about the choice of representative)

Banach space- and metric space-valued Newtonian–Sobolev functions are defined as before.

$$N^{1,p}(X : V) \qquad\qquad N^{1,p}(X : Y)$$

Recall: (X, d, μ) a metric measure space, (Y, d_Y) a metric space

Poincaré inequalities on metric measure spaces

Definition (Heinonen–Koskela)

A metric measure space (X, d, μ) **supports a p-Poincaré inequality** if there exist constants $C > 0$ and $\tau \geq 1$ s.t.

$$\fint_B |f - f_B| \, d\mu \leq C(\operatorname{diam} B) \left(\fint_{\tau B} g^p \, d\mu \right)^{1/p}$$

for all balls B and all continuous functions $f : X \to \mathbb{R}$ with upper gradient $g : X \to [0, \infty]$.

Notation: $f_B = \fint_B f = \mu(B)^{-1} \int_B f \, d\mu$.

Poincaré inequalities on metric measure spaces: remarks

$$\fint_B |f - f_B| \, d\mu \le C(\operatorname{diam} B) \left(\fint_{\tau B} g^p \, d\mu \right)^{1/p}$$

(1) PI implies good connectivity properties for X.

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(1) PI implies good connectivity properties for X.

(2) p-PI implies q-PI for $q \geq p$.

(3) If (X, d, μ) is complete and doubling and supports the p-PI, then there exists $\epsilon > 0$ s.t. (X, d, μ) supports the q-PI for $q > p - \epsilon$ (Keith–Zhong, 2008).

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(4) If X and Y are Q-regular mms with the Q-PI, then all standard definitions for quasiconformality of a homeomorphism $f : X \to Y$ are equivalent.

$$f \text{ analytically QC: } f \in N^{1,Q}(X : Y), \left(\limsup_{r \to 0} \frac{L_f(x,r)}{r} \right)^Q \le K \mu_f(x) \text{ a.e.}$$

(Heinonen–Koskela–Shanmugalingam–T, 2001)

Poincaré inequalities on metric measure spaces: applications

$$\fint_B |f - f_B| \, d\mu \le C(\operatorname{diam} B) \left(\fint_{\tau B} g^p \, d\mu \right)^{1/p}$$

(1) The **uniformization problem** for metric 2-spheres: if X is homeomorphic to $\mathbb{S}^2$, Q-regular for some $Q \ge 2$, and satisfies the Q-PI, then X is QS equiv to $\mathbb{S}^2$ (Bonk–Kleiner, 2002)

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(2) Lipschitz functions and mappings on doubling spaces supporting a PI are differentiable a.e. (Cheeger, 1999; Cheeger–Kleiner, 2000s). Applications to bi-Lipschitz nonembeddability theorems.

Poincaré inequalities on metric measure spaces: examples

1. Eucl space, compact Riem mflds, noncompact Riem mflds with curvature bounds

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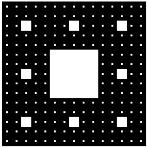

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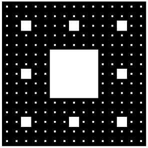

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$SC(\mathbf{a})$, $\mathbf{a} = (a_1, a_2, \ldots)$, $a_j \in \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\}$

$(SC(\mathbf{a}), d_E, \mathcal{L}^2)$ supports 1-PI iff $\mathbf{a} \in \ell^1$
$(SC(\mathbf{a}), d_E, \mathcal{L}^2)$ supports the p-PI for all $p > 1$ iff $\mathbf{a} \in \ell^2$

Sobolev–Poincaré inequalities and dimension distortion

(X, d, μ) Q-regular with a p-PI $\Rightarrow$

- $p < Q$: $N^{1,p}(X : V) \hookrightarrow L^{Qp/(Q-p)}(X : V)$,
- $p = Q$: Moser–Trudinger inequality
- $p > Q$: Morrey–Sobolev inequality ($N^{1,p} \hookrightarrow C^{0,1-Q/p}$).

$$||f(x) - f(y)|| \leq C' \, d(x,y)^{1-Q/p} \left(\int_{\tau' B} g^p \right)^{1/p}$$

if $f \in N^{1,p}(X : V)$, g is an upper gradient of f and x, y lie in a ball $B \subset X$.

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Theorem (Balogh–T–Wildrick, 2013)

Let X be proper and Ahlfors Q-regular supporting a Q-PI. Let $p > Q$. If $f : X \to Y$ is continuous and has an upper gradient in $L^p(X)$, and $E \subset X$ has $\dim E = s \in (0, Q)$, *then* $\dim f(E) \leq \frac{ps}{p-Q+s}$.

Generic dimension distortion in metric spaces

In order to formulate an analog of our generic estimates for affine subspaces, we need to understand the appropriate class of foliations in abstract metric spaces.

Definition (cf. David–Semmes)

A surjection $\pi : X \to W$ between proper metric spaces is **(locally) *s*-regular**, $s \geq 0$, if for each compact $K \subset X$, $\pi|_K$ is Lipschitz and for every ball $B \subset W$ with radius $r \leq r_0$, the (truncated) preimage $\pi^{-1}(B) \cap K$ can be covered by $\lesssim r^{-s}$ balls in X of radius $\lesssim r$.

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Examples: (1) $P_W : \mathbb{R}^n \to W$,
(2) Riem submersions $\pi : N \to N'$ (locally D–S $(\dim N - \dim N')$-regular),
(3) projection mappings in the Heisenberg group (later)

Similar to notions of **co-Lipschitz** / **Lipschitz quotient mappings**, e.g., Bates–Johnson–Lindenstrauss–Preiss–Schechtman, . . .

Theorem (Balogh–T–Wildrick, 2013)

Let X be a proper Q-regular space supporting a Q-PI. Let $\pi : X \to W$ be a David–Semmes s-regular Lipschitz surjection. Let $f : X \to Y$ be continuous with upper gradient in $L^p(X)$ for some $p > Q$. Then

$$\dim\{a \in W : \dim f(\pi^{-1}(a)) \geq \alpha\} \leq (Q - s) - p(1 - \frac{s}{\alpha})$$

for each $s < \alpha \leq \frac{ps}{p-Q+s}$.

Remarks: (1) We recover the Euclidean result in the case $X = \mathbb{R}^n$, $W = V^\perp$, $V \in G(n,m)$, $\pi = P_{V^\perp}$, $s = m$.

(2) The fibers of a D–S s-regular mapping have Hausdorff dimension $\leq s$. However, there are natural settings (e.g., certain foliations of the Heisenberg group), where this inequality is strict. In such cases, the theorem is not best possible.

The Heisenberg group $\mathbb{H}$

$\mathbb{H} = \mathbb{C} \times \mathbb{R}$ with group law

$$(z, t) * (z', t') = (z + z', t + t' + 2\,\mathrm{Im}(z\overline{z'}))$$

A basis of left-invariant vector fields is

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t, \quad T = \partial_t.$$

The **horizontal distribution** $H\mathbb{H}$ is given by $H_p\mathbb{H} = \mathrm{span}\{X(p), Y(p)\}$.

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$\mathbb{H}$ is horizontally connected by Hörmander's thm.

Carnot-Carathéodory distance $d_{cc}(\mathbf{p}, \mathbf{q})$ defined by infimizing the g_0-length of horiz curves joining $\mathbf{p}$ to $\mathbf{q}$ (g_0 is the singular Riem metric making X and Y ON).

d_{cc} is left invariant, dilations $\delta_r(z, t) = (rz, r^2 t)$ act as similarities of d_{cc}.

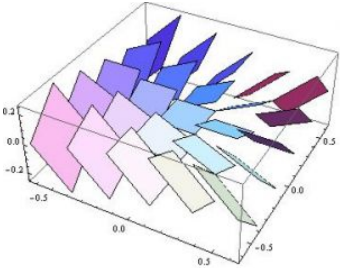
Measure on $\mathbb{H}$: Haar measure μ (agrees with $\mathcal{L}^3$ and with $\mathcal{H}^4_{cc}$ up to constants)

$(\mathbb{H}, d_{cc}, \mu)$ is Ahlfors 4-regular and supports a 1-PI.

Structure of C–C balls (**Ball-Box Theorem**): For any $\mathbf{p} \in \mathbb{H}$ and $r > 0$, $B_{cc}(\mathbf{p}, r)$ is comparable to the sheared Euclidean box

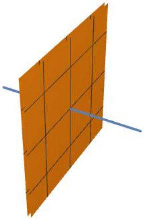
$$\mathbf{p} * [-r, r] \times [-r, r] \times [-r^2, r^2]$$

Horizontal distribution $H\mathbb{H}$



We consider foliations of $\mathbb{H}$ defined by cosets of horizontal lines and their complementary vertical planes.

Fix a horizontal line $\mathbb{V} = \mathrm{span}\{(1,0)\}$ and the complementary vertical plane $\mathbb{W} = \mathrm{span}\{(\mathbf{i},0),(0,1)\}$. These are both abelian subgroups of $\mathbb{H}$; $\mathbb{W}$ is normal.



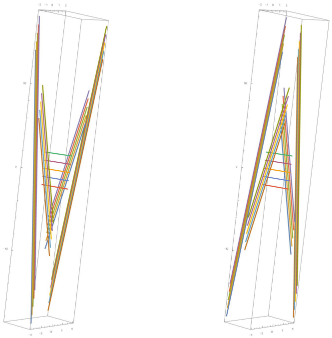
$(\mathbb{V}, d_{cc})$ is an isometrically embedded copy of $\mathbb{R}$ (so $\dim \mathbb{V} = 1$).

$d_{cc}|_{\mathbb{W}}$ is comparable to the heat metric $|y-y'| + \sqrt{|t-t'|}$ (so $\dim_{cc} \mathbb{W} = 3$).

$\mathbb{W}$ is a normal subgroup. We obtain semidirect product decompositions

$$\mathbb{H} = \mathbb{W} \ltimes \mathbb{V} \qquad \text{and} \qquad \mathbb{H} = \mathbb{V} \rtimes \mathbb{W},$$

each giving rise to a foliation of $\mathbb{H}$ by cosets of $\mathbb{V}$, parameterized by $\mathbb{W}$.



Since we are working with **left invariant** vector fields and metrics on $\mathbb{H}$, the intrinsic metric structure of these two foliations are quite different.

Left coset foliation of $\mathbb{H}$ (defined by $\mathbb{H} = \mathbb{W} \rtimes \mathbb{V}$): all fibers are 1-dimensional, parameterizing space $(\mathbb{W}, d_{cc})$ is 3-dimensional

Right coset foliation of $\mathbb{H}$ (defined by $\mathbb{H} = \mathbb{V} \rtimes \mathbb{W}$): most fibers are 2-dimensional

Since we work with a **left invariant** metric on $\mathbb{H}$, the fibers of the right coset foliation are parallel. There is a quotient metric $d_{\mathfrak{X}}$ on the right coset space $\mathfrak{X}$ s.t. $d_{\mathfrak{X}}(\mathbb{V} * a, \mathbb{V} * b) = \mathrm{dist}_{d_{\mathbb{H}}}(\mathbb{V} * a, \mathbb{V} * b)$. Then $(\mathfrak{X}, d_{\mathfrak{X}})$ is 2-dimensional and the obvious projection map $\pi^R : \mathbb{H} \to \mathfrak{X}$ is Lipschitz.

Lemma

π^R *is David–Semmes* 2*-regular.*

Corollary

$\dim\{\mathbb{V} * a \in \mathfrak{X} : \mathcal{H}^\alpha(f(\mathbb{V} * a)) > 0\} \leq 2 - p(1 - \frac{2}{\alpha})$ *for* $2 < \alpha \leq \frac{2p}{p-2}$

Intuition: The right coset foliation of $\mathbb{H}$ behaves like the foliation of $\mathbb{R}^4$ by affine translates of $V \in G(4, 2)$.

The left coset foliation of the Heisenberg group is in many ways a more natural construction. For instance, the fibers of this foliation are all horizontal (rectifiable) curves w.r.t. the sub-Riemannian metric of $\mathbb{H}$.

The left coset foliation was used by Mostow and Korányi–Reimann in their development of the theory of Heisenberg QC mappings. A key complication in Mostow's original proof of the ACL property stems from the fact that the natural projection map

$$\pi^L : (\mathbb{H}, d_{\mathbb{H}}) \to (\mathbb{W}, d_{\mathbb{H}})$$

is **not** Lipschitz.

This fact also means that the metric space theory of Sobolev and QC dimension distortion, as discussed in this lecture, cannot be applied to the left coset foliation of $\mathbb{H}$.

Nevertheless, we are able to derive some (nonsharp) dimension distortion estimates for the left coset foliation by different methods. This will be the subject of Lecture III.