A duality between Banach spaces and operators between subspaces of L_p spaces

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Fix $1 < p < \infty$ (most interesting case : $p = 2$).

The objects :

• X Banach space (not $\{0\}$).

• T bounded linear operator between sub-L_p spaces (=subspaces $E \subset L_p(\Omega_1, \mu_1)$, $F \subset L_p(\Omega_2, \mu_2)$).

The duality $\langle T, X \rangle \in \mathbb{R}$:

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The duality $\langle T, X\rangle \in \mathbb{R}$:

Define $\|T_X\|$ as the (possibly infinite) norm of $T \otimes \mathrm{id}_X$ between the subspaces $E \otimes X$ and $F \otimes X$ of $L_p(\Omega_i, \mu_i; X)$.

In formula, $\|T_X\|$ is the smallest constant C such that

$$\int_{\Omega_2} \|\sum_{i=1}^{N} T(f_i)(\omega)x_i\|_X^p d\mu_2(\omega) \leq C^p \int_{\Omega_1} \|\sum_{i=1}^{N} f_i(\omega)x_i\|_X^p d\mu_1(\omega)$$

for all N, all $f_1, \ldots, f_N \in E$ and $x_1, \ldots, x_N \in X$.

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The (vague) question :

Relate properties of T and of X to the quantity $\|T_X\|$.

Example (Fubini) : if $X \subset L_p$, then $\|T_X\| = \|T\|$. And conversely !

The polarity :
If A is a set of Banach spaces, define the *polar* of A

$$A^{\circ} = \{T, \|T_X\| \leq 1 \text{ for all } X \in A\}.$$

If B is a set of operators between sub-L_p spaces, the *polar* of B

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Question 1 (Pisier)

Describe $A^{\circ\circ}$.

By definition : $Y \in A^{\circ\circ} \iff \|T_Y\| \leq \sup_{X \in A} \|T_X\|$ for all T.

Question 2 (Pisier)

Describe $B^{\circ\circ}$.

By definition : $S \in B^{\circ\circ} \iff \|S_X\| \leq 1$ for all X s.t. $\sup_{T \in B} \|T_X\| \leq 1$.

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Example : type.
Let ε_i a sequence of iid Bernoulli random variables on a probability space (Ω, μ).
X has type > 1 (=is B-convex) if there exists $n \in N, \delta > 0$ such that

$$\|\sum \varepsilon_i x_i\|_{L_2(\Omega;X)} \leq \sqrt{n-\delta}(\sum \|x_i\|^2)^{\frac{1}{2}}.$$

(i.e X belongs to T°, where $T\colon \ell_2^n \to L_2$ maps $(x_i) \in \ell_p^n$ to $\frac{1}{\sqrt{n-\varepsilon}} \sum \varepsilon_i x_i$).

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By Pisier's theorem, this is equivalent to X being K-convex :
$\|P_X\| < \infty$, where $P\colon L_2(\Omega,\mu) \to L_2(\Omega,\mu)$ is the orthogonal projection on $\mathrm{span}\{\varepsilon_i, i \geq 1\}$. In other words, $\exists C = C(n,\varepsilon)$ such that $\frac{1}{C} P \in \{T\}^{\circ\circ}$.

Similarly for cotype :

X has cotype $q < \infty$ if there exists C such that

$$\| \sum \varepsilon_i x_i \|_{L_q(\Omega;X)} \geq C(\sum \|x_i\|^q)^{\frac{1}{q}}.$$

(i.e X belongs to T°, where $T\colon \mathrm{span}(\varepsilon_i) \subset L_q \to \ell_q$ which maps $\sum \varepsilon_i x_i$ to (x_i)).

For an integer k, define $d_k(X) \in [1, \sqrt{k}]$ by

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It is known (Milman-Wolfson 78) that the following are equivalent :

- X has type > 1.
- $\exists k$, $d_k(X) < \sqrt{k}$.
- $\lim_k \frac{d_k(X)}{\sqrt{k}} = 0$.

Question/Conjecture

In that case, there is $\beta < \frac{1}{2}$ and C such that $d_k(X) \leq Ck^\beta$.

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$$e_k(X) = \sup\{\|u_X\|, u\colon \ell_2^k \to \ell_2^k \text{ unitary}\}.$$

Then (Tomczak-Jaegerman, Pisier) $e_k(X) \leq d_k(X) \leq 2e_k(X)$. So the above question is really of the form "understand $B^{\circ\circ}$" for some B :

$$B = \{T\colon (x_i) \in \ell_2^n \to \frac{1}{\sqrt{n-\varepsilon}} \sum \varepsilon_i x_i \in L_2\}.$$

Motivation 2 : embeddability of expanders

Let $G = (V, E)$ be a finite d-regular graph, $M_G \colon \ell_2(V) \to \ell_2(V)$ the (random walk) Markov operator :

$$M_G f(x) = \frac{1}{d} \sum_{(x,y) \in E} f(y).$$

$1 = \lambda_1(G) \geq \lambda_2(G) \geq \dots \lambda_{|V|}(G)$ the eigenvalues of M_G.

Definition (Super-expanders)

A sequence $G_n = (V_n, E_n)$ of d-regular graphs is a super-expander if it is an X-pander for every superreflexive Banach space X : $\exists \gamma > 0$ such that for all n and all $f \colon V_n \to X$,

$$\frac{\gamma}{|V_n|^2} \sum_{x,y \in V_n} \|f(x) - f(y)\|^2 \leq \frac{1}{|E_n|} \sum_{(x,y) \in E_n} \|f(x) - f(y)\|^2. \qquad (1)$$

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Standard exercise : (1) for $X = \mathbb{C}$ is equivalent to $\sup_n \lambda_2(G_n) < 1$ (G_n is a sequence of expanders).

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More generally (1) for X uniformly convex is equivalent to $\sup_n \|(\tilde{M}^0_{G_n})_X\|<1$ where $\tilde{M}_G=\frac{1}{d+1}(dM_G+Id)$ is the "lazy" random walk operator and $\tilde{M}^0_G$ its restriction to $\ell_2^0=\{f\in\ell_2(V), \sum_{x\in V} f(x)=0\}$.

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Questions :

Do super-expanders exist ?

Are expanders super-expanders ?

X-valued Poincaré inequality :

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Questions :

Do super-expanders exist ? $\longrightarrow$ YES, see next slide.

Are expanders super-expanders ? $\longrightarrow$ wide open.

Theorem (Lafforgue)

Let Γ be a lattice in $\mathrm{SL}_3(\mathbb{Q}_p)$ with finite generating set S, and Γ_n a sequence of finite quotients of Γ. Then $G_n = \mathit{Cayley}(\Gamma_n, S)$ is a sequence of super-expanders (even X-panders for every K-convex space X).

Still open for lattices in $\mathrm{SL}_3(\mathbb{R})$ (eg $\mathrm{SL}_3(\mathbb{Z})$).

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Known (de Laat–dlS) : if X is such that $d_k(X) = O(x^\beta)$ for some $\beta < \frac{1}{2}$, then all lattices coming from $\mathrm{SL}_n(\mathbb{R})$ are X-panders for all $n \geq N(\beta) = O(\frac{1}{\frac{1}{2}-\beta})$.

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Conjecture (Mendel–Naor)

If X is a space with non-trivial cotype, then there is a sequence G_n of X-panders.

Motivation 3 : Group actions on Banach spaces

Recall : (Kazdhan, Delorme) Every action by affine isometries of (a lattice in) $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{Q}_p)$ ($n \geq 3$) on a Hilbert space has a fixed point.
This cannot be true if Hilbert space is replaced by arbitrary Banach space : the action of G on $\{f \in L_1(G), \int f = 1\} \sim L_1^0(G)$ has no fixed point if G is not compact.

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The conjecture holds for $\mathrm{SL}_n(\mathbb{Q}_p)$. Even, every action by affine isometries of $\mathrm{SL}_n(\mathbb{Q}_p)$ or a lattice on a K-convex space has a fixed point.

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Crucial ingredient :
(Bourgain) : if X is K-convex, there is $p > 1$ such that, for every abelian locally compact group A, its Fourier transform $\mathcal{F} \colon L_p(A) \to L_{p'}(\widehat{A})$ satisfies $\|\mathcal{F}_X\| < \infty$.

The real case.

Difficulty : harmonic analysis on $\mathrm{SO}(n)$ is not related to abelian groups ! (unlike harmonic analysis on $\mathrm{SL}_n(\mathbb{Z}_p)$, which contains "large" nilpotent groups $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$.)

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For $-1 \leq \delta \leq 1$, let $T_\delta \colon L_2(S^{n-1}) \to L_2(S^{n-1})$ the averaging operator

$$T_\delta f(x) = \text{average of } f \text{ on } \{y \in S^{n-1}, \langle x, y \rangle = \delta.$$

• (de Laat–Mimura–dlS) If X is a Banach space and there are $C, \theta > 0$ st

$$\|(T_\delta - T_0)_X\| \leq C|\delta|^\theta \ \forall \delta, \tag{2}$$

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• (de Laat–dlS) (2) holds if $d_k(X) = O(k^{\frac{1}{2} - \frac{1}{n}})$.

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Theorem (Hernandez 1983)

The bipolar of a class A of Banach spaces is the class of Banach spaces finitely representable in finite ℓ_p-direct sums of elements in A.

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For "$E^* = \{\text{operators between sub-}L_p \text{ spaces}\}$" ?

Theorem (dlS)

The bipolar of a class B of operators between sub-L_p spaces contains no other operators than the "obvious operators".

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- containing "changes of measure" operators : maps of the form $U\colon f \in L_p(\Omega,\mu) \mapsto hf \in L_p(\Omega,|h|^{-p}d\mu)$ for some $h\colon \Omega \to \mathbb{C} \setminus \{0\}$.

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- Limits of operators in B' belong to B'.

- Let $T \in B'$ such $D(T) \subset L_p(\Omega_1,m_1) \oplus L_p(\Omega,m)$ (respectively $R(T) \subset L_p(\Omega_1,m_1) \oplus L_p(\Omega,m)$) of the form $T(f\oplus g) = Sf \oplus g$ for every $f \oplus g \in D(T)$. Then S belongs to B'.

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- $S, T \in B' \implies S \oplus T \in B'$, and $\frac{1}{2}(S+T) \in B'$, $S \circ T \in B'$ when these make sense.
- Limits of operators in B' belong to B'. That is : if T is an operator between subspaces of $L_p(\Omega, m)$ and $L_p(\Omega', m')$ and if, for every finite family $f_1, \dots, f_n$ in the domain of T and every $\varepsilon > 0$, there is $S \in B'$ with domain contained in $L_p(\Omega, m)$ and range contained in $L_p(\Omega', m')$ and elements $g_1, \dots, g_n \in D(S)$ such that $\|f_i - g_i\| \leq \varepsilon$ and $\|Tf_i - Sf_i\| \leq \varepsilon$, then $T \in B'$.
- Let $T \in B'$ such $D(T) \subset L_p(\Omega_1, m_1) \oplus L_p(\Omega, m)$ (respectively $R(T) \subset L_p(\Omega_1, m_1) \oplus L_p(\Omega, m)$) of the form $T(f \oplus g) = Sf \oplus g$ for every $f \oplus g \in D(T)$. Then S belongs to B'.

The proof

Setting B set of operators between sub-L_p spaces. Define B' as in the previous slide.

We have to prove that if $T \notin B'$, there is a Banach space X such that $\|S_X\| \leq 1$ for all $S \in B$ but $\|T_X\| > 1$.

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Constructing explicit Banach spaces is a difficult task !

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Constructing explicit Banach spaces is a difficult task !

Solution : let the Hahn-Banach theorem construct X for us !

The proof

For $n \in N$, denote H_n^p the Banach space of continuous degree p homogeneous functions on $\mathbb{C}^n$:

$$H_n^p = \{\varphi \colon \mathbb{C}^n \to \mathbb{R} \text{ continuous }, \varphi(\lambda z) = |\lambda|^p \varphi(z) \forall \lambda \in \mathbb{C}, z \in \mathbb{C}^n\}.$$

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Encode a set A of Banach spaces as the cone $N(A, n) \subset H_n^p$

$$N(A, n) = \{z \mapsto \| \sum_{i=1}^{n} z_i x_i \|^p, X \in A \text{ and } x_1, \ldots, x_n \in X.\}$$

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Encode a set B of operators between sub-L_p spaces as the cone $P(B, n) \subset (H_n^p)^*$ of linear forms of the form

$$\varphi \mapsto \int \varphi(f_1(\omega), \ldots, f_n(\omega)) d\mu(\omega) - \int \varphi(Tf_1(\omega'), \ldots, Tf_n(\omega')) d\mu'(\omega')$$

for $T \in B$ and $f_1, \ldots f_n \in D(T) \subset L_p(\Omega, \mu)$.

Reason : if $\varphi(z) = \| \sum_i z_i x_i \|_X^p$,

$$\int \varphi(f_1(\omega), \ldots, f_n(\omega)) d\mu(\omega) - \int \varphi(Tf_1(\omega'), \ldots, Tf_n(\omega')) d\mu'(\omega')$$

is equal to

$$\| \sum f_i x_i \|_{L_p(\Omega; X)}^p - \| \sum Tf_i x_i \|_{L_p(\Omega'; X)}^p.$$

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So it is ≥ 0 for all $f_1, \dots, f_n \in D(T)$ iff $T \in X^\circ$.

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So it is ≥ 0 for all $f_1, \ldots, f_n \in D(T)$ iff $T \in X^\circ$.

Apply the bipolar theorem for cones in H_n^p and its dual.