MATH 409, Summer 2019, Practice Problem Set 1

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1 Natural and Real Numbers, Functions

Exercise 1. Using only the commutative field axioms of $\mathbb{R}$ prove that for all $x, y, z \in \mathbb{R}$, if $x + y = x + z$ then $y = z$.

Hint. You could use F1, F4, and F6. □

Exercise 2. Using only the commutative field axioms of $\mathbb{R}$, prove that If x is a real number, then $0 \cdot x = x \cdot 0 = 0$

Hint. You could use F1, F3, F4, and F6. □

Exercise 3. Using only the commutative field axioms of $\mathbb{R}$ and Exercise 2, prove that if x and y are real numbers so that $x \cdot y = 0$, then $x = 0$ or $y = 0$.

Hint. You could try to argue by contradiction using F1, F5, F7 and Exercise 2. □

Exercise 4. Show that for all $x, y \in \mathbb{R}$, $|xy| = |x||y|$.

Hint. Distinguish cases. □

Exercise 5. Let $x, y \in \mathbb{R}$. Show that

$$x \leqslant y \text{ if and only if } \text{ for all } \varepsilon > 0, x \leqslant y + \varepsilon.$$

Hint. You could mimic the proof of Lemma 1 in the lecture notes. □

Exercise 6. Let $x, y \in \mathbb{R}$. Show that

$$x \geqslant y \text{ if and only if } \text{ for all } \varepsilon > 0, x > y - 100\varepsilon.$$

Hint. You could mimic the proof of Lemma 1 in the lecture notes. □

Exercise 7. Let $x, y \in \mathbb{R}$. Show that if $x < (1+\varepsilon)y$ for all $\varepsilon > 0$, then $x \leqslant y$.

Hint. You could show first that y must be non-negative and then argue by contradiction. □

Exercise 8. Let $x \geqslant 0$. Show that if $x < 2\varepsilon$ for all $\varepsilon > 0$, then $x = 0$.

Hint. You could mimic the proof of 1. in Lemma 1 to show that $x \leqslant 0$ or you could prove the contrapositive. □

Exercise 9. Let $x, y \in \mathbb{R}$. Show that if for all $\varepsilon \in (0,1)$, $x \geqslant y - \varepsilon$ then $x \geqslant y$.

Hint. You could show that the seemingly weaker assumption "for all $\varepsilon \in (0,1)$, $x \geqslant y - \varepsilon$" actually implies the stronger assumption "for all $\varepsilon > 0$, $x \geqslant y - \varepsilon$". □

2 Suprema and infima

You will need the following definitions for the next exercises.

Definition 1 (Lower bound)**.** Let $X \subset \mathbb{R}$ be non-empty. A number $m \in \mathbb{R}$ (not necessarily in X) is said to be a lower bound for X if for all $x \in X$, $x \geqslant m$. A set admitting a lower bound is said to be bounded below.

Definition 2 (Infimum)**.** Let $X \subset \mathbb{R}$ be non-empty. A number $t \in \mathbb{R}$ (not necessarily in X) is called a (finite) infimum of the set X if and only if t is an lower bound for X and $t \geqslant m$ if m is any other lower bound of X.

Exercise 10. Show that if a non-empty set $X \subseteq \mathbb{R}$ has a infimum, then it is has only one infimum, that we shall denote $\inf(X)$.

Hint. You could mimic the proof of Proposition 16 in the lecture notes. $\square$

Exercise 11. Assume that a non-empty subset X of $\mathbb{R}$ has a finite infimum. Show that for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that

$$\inf(X) \leqslant x_\varepsilon < \inf(X) + \varepsilon.$$

Hint. You could mimic the proof of Lemma 2 (approximation property for suprema) in the lecture notes. $\square$

Exercise 12. Assume that a non-empty subset X of $\mathbb{Z}$ has a finite infimum. Show that $\inf(X) \in X$.

Hint. You could mimic the proof of Proposition 17 in the lecture notes. $\square$

Recall that $-E := \{x \in \mathbb{R} \colon -x \in E\}$. The results from Exercise 13 and Exercise 14 below are convenient to convert a result about suprema into a result about infima and vice-versa. They are commonly referred to as the reflection principal.

Exercise 13. Let $X \subset \mathbb{R}$ be non-empty. Show that if X has a supremum then $-X$ has an infimum, in which case

$$\inf(-X) = -\sup(X).$$

Exercise 14. Let $X \subset \mathbb{R}$ be non-empty. Show that X has a infimum then $-X$ has an supremum, in which case

$$\sup(-X) = -\inf(X).$$

Hint. You could mimic the proof of Exercise 13 $\square$

Exercise 15. Let $A \subseteq B$ be non-empty subsets of $\mathbb{R}$. Prove that if B has a supremum, then A has a supremum and $\sup(A) \leqslant \sup(B)$.

Hint. Exploit the definitions! □

Exercise 16. Let $A \subseteq B$ be non-empty subsets of $\mathbb{R}$. Prove that if B has a infimum, then A has a infimum and $\inf(B) \leqslant \inf(A)$.

Hint. You could mimic the proof of Exercise 15 or use the reflection principle together with Exercise 15. □

Exercise 17 (The greatest lower bound property). If X is a non-empty subset of $\mathbb{R}$ that is bounded below, show that X has a finite infimum in $\mathbb{R}$.

Hint. You could either mimic the proof of the least upper bound property or you could use the reflection principle and the least upper bound property. □

Exercise 18. Let A and B be non-empty subsets of $\mathbb{R}$ with the property that $a \leqslant b$ for all $a \in A$ and $b \in B$. Show that $\sup(A)$ and $\inf(B)$ exist and that $\sup(A) \leqslant \inf(B)$.

Hint. Use the least upper bound property and the greatest lower bound property. □

Exercise 19 (Cuts). Let X and Y be two subsets of $\mathbb{R}$. We say that the pair (X, Y) is a cut if

i) X and Y are non-empty

ii) $X \cup Y = \mathbb{R}$

iii) For all $x \in X$ and for all $y \in Y$ one has $x < y$.

Show the following statements about cuts:

1. If (X, Y) is a cut then $X \cap Y = \emptyset$.
2. If (X, Y) is a cut then there exists a unique $\alpha \in \mathbb{R}$ such that either

$$X = (-\infty, \alpha] \text{ and } Y = (\alpha, \infty)$$

or

$$X = (-\infty, \alpha) \text{ and } Y = [\alpha, \infty)$$

Hint. Use the Least Upper Bound Property for (2). $\square$