MATH 409, Summer 2019, Practice Problem Set 1

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1 Natural and Real Numbers, Functions

Exercise 1. Using only the commutative field axioms of $\mathbb{R}$ prove that for all $x, y, z \in \mathbb{R}$, if $x + y = x + z$ then $y = z$.

Hint. You could use F1, F4, and F6. □

Possible solution. Assume that $x+y = x+z$. Since x admits an additive inverse $-x$ one has $-x + (x + y) = -x + (x + z)$ and by associativity of the addition $(-x + x) + y = (-x + x) + z$. By the property of the additive inverse it follows that $0 + y = 0 + z$, and thus $y = z$ by F4. □

Exercise 2. Using only the commutative field axioms of $\mathbb{R}$, prove that If x is a real number, then $0 \cdot x = x \cdot 0 = 0$

Hint. You could use F1, F3, F4, and F6. □

Possible solution. Note that by using F4 and F3, on has $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$. But by F6 every element has an additive inverse and denote z the additive inverse of $0 \cdot x$, i.e. $z + 0 \cdot x = 0 \cdot x + z = 0$. Then $z + 0 \cdot x = z + (0 \cdot x + 0 \cdot x)$, and thus by F1 $z + 0 \cdot x = (z + 0 \cdot x) + 0 \cdot x$. And hence $0 = 0 + 0 \cdot x = 0 \cdot x$ using F4 one more time. A similar argument works to show that $x \cdot 0 = 0$. □

Exercise 3. Using only the commutative field axioms of $\mathbb{R}$ and Exercise 2, prove that if x and y are real numbers so that $x \cdot y = 0$, then $x = 0$ or $y = 0$.

Hint. You could try to argue by contradiction using F1, F5, F7 and Exercise 2. □

Possible solution. By contradiction assume that $x \cdot y = 0$ with $x \neq 0$ and $y \neq 0$. Since x is non-zero, x has a multiplicative inverse x^{-1} by F7 and $y = 1 \cdot y = (x^{-1} \cdot x) \cdot y = x^{-1} \cdot (x \cdot y) = x^{-1} \cdot 0 = 0$, where for the first equality we used F5, for the third F1, and for the last one Exercise 2. But this contradicts the assumption that $y \neq 0$. □

Exercise 4. Show that for all $x, y \in \mathbb{R}$, $|xy| = |x||y|$.

Hint. Distinguish cases. □

Exercise 5. Let $x, y \in \mathbb{R}$. Show that

$$x \leqslant y \text{ if and only if } \text{ for all } \varepsilon > 0, x \leqslant y + \varepsilon.$$

Hint. You could mimic the proof of Lemma 1 in the lecture notes. □

Possible solution. If $x \leqslant y$ then clearly $x \leqslant y + \varepsilon$ whenever $\varepsilon > 0$. Assume now that for all $\varepsilon > 0$, $x \leqslant y + \varepsilon$. Assume for the sake of a contradiction that $x > y$ and let $\varepsilon = 2(x - y)$. Then $x \leqslant y + \varepsilon = y + 2(x - y) = 2x - y$, and thus $y \leqslant x$, a contradiction. □

Exercise 6. Let $x, y \in \mathbb{R}$. Show that

$$x \geqslant y \text{ if and only if } \text{ for all } \varepsilon > 0, x > y - 100\varepsilon.$$

Hint. You could mimic the proof of Lemma 1 in the lecture notes. □

Possible solution. The direct implication is elementary. Indeed, ff $x \geqslant y$ then clearly $x \geqslant y - 100\varepsilon$ whenever $\varepsilon > 0$. For the converse, assume that $x > y - \varepsilon$ for all $\varepsilon > 0$. Assume by contradiction that $y > x$ and let $\varepsilon_0 = \frac{y-x}{100} > 0$. By our assumption, $x > y - 100\varepsilon_0 = y - (y - x) = x$; a contradiction. □

Exercise 7. Let $x, y \in \mathbb{R}$. Show that if $x < (1 + \varepsilon)y$ for all $\varepsilon > 0$, then $x \leqslant y$.

Hint. You could show first that y must be non-negative and then argue by contradiction. □

Possible solution. Let $x, y \in \mathbb{R}$ and assume that for all $\varepsilon > 0$, $x < (1+\varepsilon)y$. Remark first that $y \geqslant 0$. Assume by contradiction that $y < 0$, then for all $n \geqslant 2$, $x < ny$ (simply take $\varepsilon = n-1$) and $\frac{x}{y} > n$, which implies that $\mathbb{N}$ is bounded; a contradiction. Therefore, we may assume that $x \in \mathbb{R}$, $y \geqslant 0$ and for all $\varepsilon > 0$, $x < (1+\varepsilon)y$. If $y = 0$ the conclusion immediately holds. If $y > 0$ then $x < y + \varepsilon y$ and $\frac{x}{y} < 1 + \varepsilon$, for every $\varepsilon > 0$. And hence, $\frac{x}{y} \leqslant 1$ and $x \leqslant y$. □

Exercise 8. Let $x \geqslant 0$. Show that if $x < 2\varepsilon$ for all $\varepsilon > 0$, then $x = 0$.

Hint. You could mimic the proof of 1. in Lemma 1 to show that $x \leqslant 0$ or you could prove the contrapositive. □

Possible solution. Assume that $x \geqslant 0$. We will prove the contrapositive. Assume that $x \neq 0$, then $x > 0$. If we let $\varepsilon_0 = \frac{x}{2} > 0$ then $2\varepsilon_0 = x \leqslant x$; the contrapositive is proven. □

Exercise 9. Let $x, y \in \mathbb{R}$. Show that if for all $\varepsilon \in (0,1)$, $x \geqslant y - \varepsilon$ then $x \geqslant y$.

Hint. You could show that the seemingly weaker assumption "for all $\varepsilon \in (0,1)$, $x \geqslant y - \varepsilon$" actually implies the stronger assumption "for all $\varepsilon > 0$, $x \geqslant y - \varepsilon$". □

Possible solution. Let $x, y \in \mathbb{R}$ and assume that for all $\varepsilon \in (0,1)$, $x \geqslant y - \varepsilon$. Let $\varepsilon \geqslant 1$ then $y - \varepsilon \leqslant y - 1 \leqslant y - \frac{1}{2} \leqslant x$, where in the last inequality we use our assumption. Therefore for all $\varepsilon > 0$, $x \geqslant y - \varepsilon$. Now an argument similar to the one in Exercise 5 shows that $x \geqslant y$. □

2 Suprema and infima

You will need the following definitions for the next exercises.

Definition 1 (Lower bound)**.** Let $X \subset \mathbb{R}$ be non-empty. A number $m \in \mathbb{R}$ (not necessarily in X) is said to be a lower bound for X if for all $x \in X$, $x \geqslant m$. A set admitting a lower bound is said to be bounded below.

Definition 2 (Infimum)**.** Let $X \subset \mathbb{R}$ be non-empty. A number $t \in \mathbb{R}$ (not necessarily in X) is called a (finite) infimum of the set X if and only if t is an lower bound for X and $t \geqslant m$ if m is any other lower bound of X.

Exercise 10. Show that if a non-empty set $X \subseteq \mathbb{R}$ has a infimum, then it is has only one infimum, that we shall denote $\inf(X)$.

Hint. You could mimic the proof of Proposition 16 in the lecture notes. □

Possible solution. Assume that X has two infima t_1 and t_2. Then by definition t_1 is a lower bound but t_2 being an infimum we have $t_1 \leqslant t_2$. A similar argument, tells us that $t_2 \leq t_1$, and we conclude by antisymmetry of the order relation that $t_1 = t_2$. □

Exercise 11. Assume that a non-empty subset X of $\mathbb{R}$ has a finite infimum. Show that for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that

$$\inf(X) \leqslant x_\varepsilon < \inf(X) + \varepsilon.$$

Hint. You could mimic the proof of Lemma 2 (approximation property for suprema) in the lecture notes. □

Possible solution. The left-hand side inequality holds for every element in X by definition of the infimum and only the right-hand side inequality requires a proof. Assume by contradiction that there exists $\varepsilon_0 > 0$ such that for all $x \in X$, $\inf(X) + \varepsilon_0 \leqslant x$, and thus $\inf(X) + \varepsilon_0$ is a lower bound for X that is strictly larger than $\inf(X)$; a contradiction. □

Exercise 12. Assume that a non-empty subset X of $\mathbb{Z}$ has a finite infimum. Show that $\inf(X) \in X$.

Hint. You could mimic the proof of Proposition 17 in the lecture notes. □

Recall that $-E := \{x \in \mathbb{R} \colon -x \in E\}$. The results from Exercise 13 and Exercise 14 below are convenient to convert a result about suprema into a result about infima and vice-versa. They are commonly referred to as the reflection principal.

Exercise 13. Let $X \subset \mathbb{R}$ be non-empty. Show that if X has a supremum then $-X$ has an infimum, in which case

$$\inf(-X) = -\sup(X).$$

Possible solution. Assume that E has a supremum $s = \sup(E)$, we will show that $t = -s$ is the infimum of $-E$. Indeed t is a lower bound for E since for all $x \in E$, $-x \in -E$ and $t \leqslant x$ follows from $-x \leqslant -t = s$. Assume now that l is another lower bound for $-E$ then $-l$ is an upper bound for E and $s \leqslant -l$, and thus $l \leqslant -s = t$ and the implication follows. □

Exercise 14. Let $X \subset \mathbb{R}$ be non-empty. Show that X has a infimum then $-X$ has an supremum, in which case

$$\sup(-X) = -\inf(X).$$

Hint. You could mimic the proof of Exercise 13 $\square$

Exercise 15. Let $A \subseteq B$ be non-empty subsets of $\mathbb{R}$. Prove that if B has a supremum, then A has a supremum and $\sup(A) \leqslant \sup(B)$.

Hint. Exploit the definitions! $\square$

Possible solution. If $s = \sup(B)$ then for all $b \in B$, $b \leqslant s$. Since $A \subseteq B$, for every $a \in A$, $a \leqslant s$ and s is an upper bound for A. By the least upper bound property $\sup(A)$ exists and $\sup(A) \leqslant s$ by the definition of the supremum. $\square$

Exercise 16. Let $A \subseteq B$ be non-empty subsets of $\mathbb{R}$. Prove that if B has a infimum, then A has a infimum and $\inf(B) \leqslant \inf(A)$.

Hint. You could mimic the proof of Exercise 15 or use the reflection principle together with Exercise 15. $\square$

Possible solution. A proof similar to the one above will work or you can use the reflection principle as follows. If $A \subset B$ then $-A \subset -B$ (prove it!). If B has an infimum, then $-B$ has a supremum and $\inf(B) = -\sup(-B) \leqslant -\sup(-A) = \inf(A)$. $\square$

Exercise 17 (The greatest lower bound property). If X is a non-empty subset of $\mathbb{R}$ that is bounded below, show that X has a finite infimum in $\mathbb{R}$.

Hint. You could either mimic the proof of the least upper bound property or you could use the reflection principle and the least upper bound property. $\square$

Possible solution. Assume that E is bounded below then $-E$ is bounded above and has a supremum by the least upper bound property. It follows from the reflection principle that E has an infimum. $\square$

Exercise 18. Let A and B be non-empty subsets of $\mathbb{R}$ with the property that $a \leqslant b$ for all $a \in A$ and $b \in B$. Show that $\sup(A)$ and $\inf(B)$ exist and that $\sup(A) \leqslant \inf(B)$.

Hint. Use the least upper bound property and the greatest lower bound property. □

Possible solution. Fix b in B. Then by assumption, for all a in A we have $a \leqslant b$, i.e. b is an upper bound for A. By the least upper bound property, $\sup(A)$ exists, and since every $b \in B$ is an upper bound for A, we have $\sup(A) \leqslant b$ for all $b \in B$. Therefore $\sup(A)$ is a lower bound for B. By the greatest lower bound property, $\inf(B)$ exists, and since $\sup(A)$ is a lower bound for B, we have $\sup(A) \leqslant \inf(B)$. □

Exercise 19 (Cuts). Let X and Y be two subsets of $\mathbb{R}$. We say that the pair (X, Y) is a cut if

i) X and Y are non-empty

ii) $X \cup Y = \mathbb{R}$

iii) For all $x \in X$ and for all $y \in Y$ one has $x < y$.

Show the following statements about cuts:

1. If (X, Y) is a cut then $X \cap Y = \emptyset$.
2. If (X, Y) is a cut then there exists a unique $\alpha \in \mathbb{R}$ such that either

$$X = (-\infty, \alpha] \text{ and } Y = (\alpha, \infty)$$

or

$$X = (-\infty, \alpha) \text{ and } Y = [\alpha, \infty)$$

Hint. Use the Least Upper Bound Property for (2). □