MATH 409, Summer 2019,
Practice Problem Set 2

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1 Warm up

Exercise 1. Show that a sequence $(x_n)_{n=1}^{\infty}$ is convergent to $\ell \in \mathbb{R}$, if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|x_n - \ell| \leqslant \varepsilon$.

Hint. Exploit the definition. □

Exercise 2. Show that a sequence $(x_n)_{n=1}^{\infty}$ is convergent to $\ell \in \mathbb{R}$, if and only if for every $\varepsilon \in (0,2)$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|x_n - \ell| < \varepsilon$.

Hint. Exploit the definition. □

Exercise 3. Show that a sequence $(x_n)_{n=1}^{\infty}$ is convergent to $\ell \in \mathbb{R}$, if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|x_n - \ell| < 256\varepsilon$.

Hint. Exploit the definition. □

Exercise 4. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be convergent sequences. Show that

1. the sequence $(x_n + y_n)_{n=1}^{\infty}$ is convergent and that

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

2. the sequence $(2x_n - 5y_n)_{n=1}^{\infty}$ is convergent and that

$$\lim_{n\to\infty}(2x_n - 5y_n) = 2\lim_{n\to\infty} x_n - 5\lim_{n\to\infty} y_n.$$

3. the sequence $(x_n \cdot y_n)_{n=1}^{\infty}$ is convergent and that

$$\lim_{n\to\infty} (x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n.$$

Hint. Use the definition of convergence and the algebraic equality, $ab - cd = b(a-c) + c(b-d)$ (for 3) □

Exercise 5. Show that $(\frac{1}{3^n})_{n=1}^{\infty}$ converges and compute $\lim\limits_{n\to\infty} \dfrac{1}{3^n}$.

Hint. Try to use the idea of the proof of 3. in Example 1. □

2 Useful results about sequences

Exercise 6. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $\ell \in \mathbb{R}$. Show that $\lim_{n\to\infty} x_n = \ell$ if and only if $\lim_{n\to\infty} |x_n - \ell| = 0$.

Hint. Simply consider the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty} = (|x_n - \ell|)_{n=1}^{\infty}$, and apply the definition of convergence. □

Exercise 7. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences of real numbers and $\ell \in \mathbb{R}$. Assume that $\lim_{n\to\infty} x_n = 0$ and that there exists $N \in \mathbb{N}$ so that for all $n \geqslant N$ we have $|y_n - \ell| \leqslant |x_n|$. Show that $\lim_{n\to\infty} y_n = \ell$.

Hint. Exploit the definition of convergence. □

Exercise 8. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences of real numbers. Assume that $(x_n)_{n=1}^{\infty}$ is bounded and that $\lim_{n\to\infty} y_n = 0$. Let $(z_n)_{n=1}^{\infty} = (x_n \cdot y_n)_{n=1}^{\infty}$. Show that $\lim_{n\to\infty} z_n = 0$

Hint. Exploit the definitions of convergence, boundedness, and the properties of the absolute value. □

Exercise 9. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Show that $(x_n)_{n=1}^{\infty}$ is increasing if and only if for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$.

Hint. One implication follows directly from the definition. The other one can be proven using an induction. □

3 Around the Monotone Convergence Theorem

Exercise 10. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Show without using the Monotone Convergence Theorem that if $(x_n)_{n=1}^{\infty}$ is decreasing and bounded below then $(x_n)_{n=1}^{\infty}$ is convergent.

Hint. You could mimic the proof of the increasing version and use the approximation property of infima to show that $(x_n)_{n=1}^{\infty}$ converges to $\inf\{x_n : n \in \mathbb{N}\}$. □

Exercise 11. Show that if $|a| < 1$ then $\lim_{n\to\infty} a^n = 0$.

Hint. Use the Monotone Convergence Theorem. □

Exercise 12. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. For all $n \in \mathbb{N}$, let $t_n := \inf\{x_k : k \geq n\}$. Show that $(t_n)_{n=1}^{\infty}$ is convergent.

Hint. You could use the Monotone Convergent Theorem and mimic the proof of Lemma 7 in the lecture notes. □

Exercise 13 (The Nested Interval Theorem). Recall that a sequence of set $(A_n)_{n\in\mathbb{N}}$ is nested if for all $n \in \mathbb{N}$, $A_{n+1} \subseteq A_n$. Recall also that a closed interval is a subset of $\mathbb{R}$ of the form $[a, b]$. Show that a nested sequence of closed intervals has a non-empty intersection.

Hint. You could use the Least Upper Bound Theorem or the Monotone Convergent Theorem. □

Exercise 14. Let $0 < x_1 < y_1$ and set for all $n \in \mathbb{N}$,

$$x_{n+1} = \sqrt{x_n y_n} \text{ and } y_{n+1} = \frac{x_n + y_n}{2}.$$

i) Prove that for all $n \in \mathbb{N}$, $0 < x_n < y_n$.

ii) Prove that $(x_n)_{n=1}^{\infty}$ is increasing and bounded above.

iii) Prove that $(y_n)_{n=1}^{\infty}$ is decreasing and bounded below.

iv) Prove that for all $n \in \mathbb{N}$, $0 < y_{n+1} - x_{n+1} < \frac{y_1 - x_1}{2^n}$.

v) Prove that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

This common limit $\alpha := \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ in v) above is called the arithmetic-geometric mean of x_1 and y_1 and has many applications.

Hint. For i) use induction. For ii)-iii) use i). For iv) use induction. For v) use the Monotone Convergence Theorem and the Squeeze Theorem. □

4 Subsequences

Exercise 15. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $a \in \mathbb{R}$. If the sequence $(x_n)_{n=1}^{\infty}$ does not converge to a, prove that there exists an $\varepsilon_0 > 0$ and a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, so that $|x_{n_k} - a| \geqslant \varepsilon_0$ for all $k \in \mathbb{N}$.

Hint. Negate the definition of convergence and construct the subsequence recursively. □

Exercise 16. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $\ell \in \mathbb{R}$. Assume that for every subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, there exists a further subsequence $(z_n)_{n=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ that converges to ℓ. Prove that the original sequence $(x_n)_{n=1}^{\infty}$ converges to ℓ.

Hint. Argue by contradiction using Exercise 15. □

Exercise 17. For this exercise we will define a top point of a sequence $(x_n)_{n=1}^{\infty}$ as follows: we say that x_p is a top point of the sequence if for all $n \geq p$, $x_n \leq x_p$. Prove the monotone subsequence lemma using the notion of top point.

Hint. Consider the following three cases: the sequence has infinitely many top points, or finitely many top points, or no top points. □

Exercise 18. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let $t := \liminf_{n\to\infty} x_n$. Show that there exists a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_n = t$.

Hint. Construct the subsequence recursively using the approximation property for suprema and conclude with the Squeeze Theorem. □

5 Constructing sequences

Exercise 19. Prove that for every real number x there exists a sequence of rational numbers $(q_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} q_n = x$.

Hint. Use the density of $\mathbb{Q}$ in $\mathbb{R}$ and the Squeeze Theorem. □

Exercise 20. Let X be a non-empty subset of $\mathbb{R}$ that is bounded above. Assume that $\sup(X) \notin X$. Prove that there exists a strictly increasing sequence $(x_n)_{n=1}^{\infty}$ of X so that $\lim_{n\to\infty} x_n = \sup(X)$.

Hint. Construct the sequence recursively using the approximation property for suprema. □

Exercise 21. Let X be a non-empty subset of $\mathbb{R}$ that is bounded below. Assume that $\inf(X) \notin X$. Prove that there exists a strictly decreasing sequence $(x_n)_{n=1}^{\infty}$ of X so that $\lim_{n\to\infty} x_n = \inf(X)$.

Hint. Mimic the proof of Exercise 20. □

6 Cauchy Sequences

Exercise 22. Show that a Cauchy sequence is bounded.

Hint. The proof is similar to the proof of the fact that a convergent sequence is bounded. □

Exercise 23. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two Cauchy sequences such that $|y_n| \geq \alpha > 0$ for all $n \in \mathbb{N}$. Show that the sequence $(\frac{x_n}{y_n})_{n=1}^{\infty}$ is Cauchy.

Hint. Use the Triangle Inequality and ad-hoc algebraic manipulations. □

Exercise 24. Let $(x_n)_{n=1}^{\infty}$ be a sequence of integers, i.e. $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$.

(i) If $(x_n)_{n=1}^{\infty}$ is Cauchy, show that it is eventually constant (i.e. there exists $n_0 \in \mathbb{N}$ so that for all $n \geqslant n_0$ we have $x_n = x_{n_0}$).

(ii) If $(x_n)_{n=1}^{\infty}$ converges to some $\ell \in \mathbb{R}$, then $\ell \in \mathbb{Z}$.

Hint. For i) use the definition of Cauchy sequence for a well chosen ε and derive a contradiction if the sequence is not eventually constant. For ii) use i). $\square$

Exercise 25. Let $(x_n)_{n=1}^{\infty}$ be a sequence. Suppose that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m \geq n \geq N$, $|\sum_{k=n}^{m} x_k| < \varepsilon$. Prove that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} x_k$ exists and is finite.

Hint. If you introduce a well chosen sequence it is a one line argument. $\square$

Exercise 26. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq \frac{1}{3^n}$. Show that $(x_n)_{n=1}^{\infty}$ is convergent.

Hint. Show that $(x_n)_{n=1}^{\infty}$ is Cauchy. $\square$