MATH 409, Summer 2019, Practice Problem Set 2

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1 Warm up

Exercise 1. Show that a sequence $(x_n)_{n=1}^{\infty}$ is convergent to $\ell \in \mathbb{R}$, if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|x_n - \ell| \leqslant \varepsilon$.

Hint. Exploit the definition. □

Possible solution. □

Exercise 2. Show that a sequence $(x_n)_{n=1}^{\infty}$ is convergent to $\ell \in \mathbb{R}$, if and only if for every $\varepsilon \in (0, 2)$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|x_n - \ell| < \varepsilon$.

Hint. Exploit the definition. □

Possible solution. □

Exercise 3. Show that a sequence $(x_n)_{n=1}^{\infty}$ is convergent to $\ell \in \mathbb{R}$, if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|x_n - \ell| < 256\varepsilon$.

Hint. Exploit the definition. □

Possible solution. □

Exercise 4. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be convergent sequences. Show that

1. the sequence $(x_n + y_n)_{n=1}^{\infty}$ is convergent and that

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

2. the sequence $(2x_n - 5y_n)_{n=1}^{\infty}$ is convergent and that

$$\lim_{n\to\infty}(2x_n - 5y_n) = 2\lim_{n\to\infty} x_n - 5\lim_{n\to\infty} y_n.$$

3. the sequence $(x_n \cdot y_n)_{n=1}^{\infty}$ is convergent and that

$$\lim_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n.$$

Hint. Use the definition of convergence and the algebraic equality, $ab - cd = b(a-c) + c(b-d)$ (for 3) □

Possible solution. Assume that $\lim_{n\to\infty} x_n = \ell_1 < \infty$ and $\lim_{n\to\infty} y_n = \ell_2 < \infty$.

1. Let $\varepsilon > 0$, then there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$, $|x_n - \ell_1| < \frac{\varepsilon}{2}$ and for all $n \geq N_2$, $|y_n - \ell_2| < \frac{\varepsilon}{2}$. If follows from the triangle inequality that $|x_n + y_n - (\ell_1 + \ell_2)| = |x_n - \ell_1 + y_n - \ell_2| \leq |x_n - \ell_1| + |y_n - \ell_2|$, and hence for $n \geq \max\{N_1, N_2\}$, $|x_n + y_n - (\ell_1 + \ell_2) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

2. If follows from the triangle inequality that $|x_n \cdot y_n - (\ell_1 \cdot \ell_2)| = |(x_n - \ell_1)y_n + \ell_1(y_n - \ell_2)| \leq |x_n - \ell_1||y_n| + |y_n - \ell_2||\ell_1|$. Since $(y_n)_{n=1}^{\infty}$ is convergent, and thus bounded, there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|y_n| \leq M$. Let $\varepsilon > 0$. If $|\ell_1| > 0$, then there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$, $|x_n - \ell_1| < \frac{\varepsilon}{2M}$ and for all $n \geq N_2$, $|y_n - \ell_2| < \frac{\varepsilon}{2|\ell_1|}$, and hence for $n \geq \max\{N_1, N_2\}$, $|x_n \cdot y_n - (\ell_1 \cdot \ell_2)| < \frac{\varepsilon}{2M}M + \frac{\varepsilon}{2|\ell_1|}|\ell_1| = \varepsilon$. If $|\ell_1| = 0$ then for $n \geq \max\{N_1, N_2\}$, $|x_n \cdot y_n| < \frac{\varepsilon}{2M}M < \varepsilon$, and the proof is complete.

□

Exercise 5. Show that $(\frac{1}{3^n})_{n=1}^{\infty}$ converges and compute $\lim\limits_{n\to\infty} \dfrac{1}{3^n}$.

Hint. Try to use the idea of the proof of 3. in Example 1. □

Possible solution. It follows from the Archimedean Principle that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $0 < \frac{1}{\varepsilon} < N$. It can be easily shown by induction (do it!) that $n \leq 3^n$ for all $n \in \mathbb{N}$. For $n \geq N$, $|\frac{1}{3^n} - 0| = \frac{1}{3^n} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ and $\lim_{n\to\infty} \frac{1}{3^n} = 0$. □

2 Useful results about sequences

Exercise 6. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $\ell \in \mathbb{R}$. Show that $\lim_{n\to\infty} x_n = \ell$ if and only if $\lim_{n\to\infty} |x_n - \ell| = 0$.

Hint. Simply consider the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty} = (|x_n - \ell|)_{n=1}^{\infty}$, and apply the definition of convergence. $\square$

Possible solution. Assume that $\lim_{n\to\infty} x_n = \ell$. Let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - \ell| < \varepsilon$. But $\big||x_n - \ell| - 0\big| = |x_n - \ell|$ and for $n \geq N$, $\big||x_n - \ell| - 0\big| < \varepsilon$, and thus $\lim_{n\to\infty} |x_n - \ell| = 0$.

Assume now that $\lim_{n\to\infty} |x_n - \ell| = 0$, Let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\big||x_n - \ell| - 0\big| < \varepsilon$ but $\big||x_n - \ell| - 0\big| = |x_n - \ell|$ and for $n \geq N$, $|x_n - \ell| < \varepsilon$. Therefore, $\lim_{n\to\infty} x_n = \ell$. $\square$

Exercise 7. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences of real numbers and $\ell \in \mathbb{R}$. Assume that $\lim_{n\to\infty} x_n = 0$ and that there exists $N \in \mathbb{N}$ so that for all $n \geqslant N$ we have $|y_n - \ell| \leqslant |x_n|$. Show that $\lim_{n\to\infty} y_n = \ell$.

Hint. Exploit the definition of convergence. $\square$

Possible solution. Assume that $\lim_{n\to\infty} x_n = 0$ and that there exists $N \in \mathbb{N}$ so that for all $n \geqslant N$ we have $|y_n - \ell| \leqslant |x_n|$. Let $\varepsilon > 0$. Then there $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|x_n| < \varepsilon$. If $n \geq \max\{N, N_1\}$ then, $|y_n - \ell| \leq |x_n| < \varepsilon$. Therefore, $\lim_{n\to\infty} y_n = \ell$. $\square$

Exercise 8. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences of real numbers. Assume that $(x_n)_{n=1}^{\infty}$ is bounded and that $\lim_{n\to\infty} y_n = 0$. Let $(z_n)_{n=1}^{\infty} = (x_n \cdot y_n)_{n=1}^{\infty}$. Show that $\lim_{n\to\infty} z_n = 0$

Hint. Exploit the definitions of convergence, boundedness, and the properties of the absolute value. $\square$

Possible solution. Assume that there exists $M \geq 0$ such that for all $n \in \mathbb{N}$, $|x_n| \leq M$ and that $\lim_{n\to\infty} y_n = 0$. If $M = 0$ $|x_n y_n| = 0$ and the conclusion clearly holds. Otherwise, if $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|y_n| < \frac{\varepsilon}{M}$. This yields that $|x_n y_n| = |x_n||y_n| \leq M|y_n| < M\frac{\varepsilon}{M} = \varepsilon$ whenever $n \geq N$. Therefore, $(z_n)_{n=1}^{\infty}$ converges to 0. $\square$

Exercise 9. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Show that $(x_n)_{n=1}^{\infty}$ is increasing if and only if for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$.

Hint. One implication follows directly from the definition. The other one can be proven using an induction. □

Possible solution. Assume that $(x_n)_{n=1}^{\infty}$ is increasing, i.e. for all $k \leq m$, $x_k \leq x_m$. Let $n \in \mathbb{N}$, then by simply taking $k = n$ and $m = n+1$, one has $x_n \leq x_{n+1}$.

Assume now that for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$. Since when $k < m$ one can always write $m = k + r$ for some $r \in \mathbb{N}$, the conclusion will follow if one can prove that for all $k, r \in \mathbb{N}$, $x_k \leq x_{k+r}$. Let $k \in \mathbb{N}$ and for $r \in \mathbb{N}$ let $P(r)$ be the statement: $x_k \leq x_{k+r}$. Our assumption says that $P(1)$ is true. Assume now that $P(r)$ is true. On one hand, $x_k \leq x_{k+r}$ by our induction hypothesis. On the other hand, $x_{k+r} \leq x_{k+r+1}$ by our assumption, and hence by transitivity of the order relation $x_k \leq x_{k+r+1}$ and $P(r+1)$ is true. By the Principle of Mathematical Induction $P(r)$ is true for all $r \in \mathbb{N}$. Since k was fixed but arbitrary, one just proved that for all $k, r \in \mathbb{N}$, $x_k \leq x_{k+r}$ and the conclusion follows. □

3 Around the Monotone Convergence Theorem

Exercise 10. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Show without using the Monotone Convergence Theorem that if $(x_n)_{n=1}^{\infty}$ is decreasing and bounded below then $(x_n)_{n=1}^{\infty}$ is convergent.

Hint. You could mimic the proof of the increasing version and use the approximation property of infima to show that $(x_n)_{n=1}^{\infty}$ converges to $\inf\{x_n : n \in \mathbb{N}\}$. □

Possible solution. □

Exercise 11. Show that if $|a| < 1$ then $\lim\limits_{n\to\infty} a^n = 0$.

Hint. Use the Monotone Convergence Theorem. □

Possible solution. Assume that $|a| < 1$, then $|a|^{n+1} = |a|^n \cdot a < |a|^n$ and $(|a|^n)_{n=1}^{\infty}$ is strictly decreasing. It is clear that $(|a|^n)_{n=1}^{\infty}$ is bounded below by 0, and by the Monotone Convergence Theorem, $(|a|^n)_{n=1}^{\infty}$ is convergent. Denote ℓ the limit. Then $\lim_{n\to\infty} |a|^n = \lim_{n\to\infty} |a|^{n+1} = \ell$ and since $|a|^{n+1} = |a|^n \cdot a$ by the basic manipulations of limits ℓ satisfies the equation $\ell = \ell \cdot |a|$. The only solutions are $\ell = 0$ or $|a| = 1$ and the second alternative is impossible, thus $\lim_{n\to\infty} |a|^n = 0$. We conclude with the Squeeze Theorem since for all $n \in \mathbb{N}$, $-|a|^n \leq a^n \leq |a|^n$. □

Exercise 12. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. For all $n \in \mathbb{N}$, let $t_n := \inf\{x_k \colon k \geq n\}$. Show that $(t_n)_{n=1}^{\infty}$ is convergent.

Hint. You could use the Monotone Convergent Theorem and mimic the proof of Lemma 7 in the lecture notes. □

Possible solution. Let $n \in \mathbb{N}$. Since $\{x_k \colon k \geq n\} \supset \{x_k \colon k \geq n+1\}$, $t_n = \inf\{x_k \colon k \geq n\} \leq \inf\{x_k \colon k \geq n+1\} = t_{n+1}$, and $(t_n)_{n=1}^{\infty}$ is increasing. Since $(x_n)_{n=1}^{\infty}$ is bounded, $(t_n)_{n=1}^{\infty}$ is also bounded. By the Monotone Convergence Theorem $(t_n)_{n=1}^{\infty}$ is convergent. □

Exercise 13 (The Nested Interval Theorem). Recall that a sequence of set $(A_n)_{n\in\mathbb{N}}$ is nested if for all $n \in \mathbb{N}$, $A_{n+1} \subseteq A_n$. Recall also that a closed interval is a subset of $\mathbb{R}$ of the form $[a, b]$. Show that a nested sequence of closed intervals has a non-empty intersection.

Hint. You could use the Least Upper Bound Theorem or the Monotone Convergent Theorem. □

Possible solution. □

Exercise 14. Let $0 < x_1 < y_1$ and set for all $n \in \mathbb{N}$,

$$x_{n+1} = \sqrt{x_n y_n} \text{ and } y_{n+1} = \frac{x_n + y_n}{2}.$$

i) Prove that for all $n \in \mathbb{N}$, $0 < x_n < y_n$.

ii) Prove that $(x_n)_{n=1}^{\infty}$ is increasing and bounded above.

iii) Prove that $(y_n)_{n=1}^{\infty}$ is decreasing and bounded below.

iv) Prove that for all $n \in \mathbb{N}$, $0 < y_{n+1} - x_{n+1} < \frac{y_1 - x_1}{2^n}$.

v) Prove that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

This common limit $\alpha := \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ in v) above is called the arithmetic-geometric mean of x_1 and y_1 and has many applications.

Hint. For i) use induction. For ii)-iii) use i). For iv) use induction. For v) use the Monotone Convergence Theorem and the Squeeze Theorem. □

Possible solution. (i) For $n = 1$ the inequality holds by assumption. If $n \geq 2$, then

$$x_n = \sqrt{x_{n-1}y_{n-1}} \text{ and } y_n = \frac{x_{n-1} + y_{n-1}}{2},$$

and one needs to show that the geometric mean is smaller that the arithmetic mean i.e. $\sqrt{x_{n-1}y_{n-1}} < \frac{x_{n-1}+y_{n-1}}{2}$ or equivalently, $(x_{n-1}+y_{n-1})^2 > 4x_{n-1}y_{n-1}$. But,

$$\begin{aligned}(x_{n-1} + y_{n-1})^2 - 4x_{n-1}y_{n-1} &= x_{n-1}^2 + 2x_{n-1}y_{n-1} + y_{n-1}^2 - 4x_{n-1}y_{n-1} \\ &= x_{n-1}^2 - 2x_{n-1}y_{n-1} + y_{n-1}^2 \\ &= (x_{n-1} - y_{n-1})^2 \geq 0.\end{aligned}$$

The conclusion follows since one can easily prove by induction (and we omit the details) that for all $n \in \mathbb{N}$, $x_n, y_n > 0$ and $x_n \neq y_n$.

(ii) By (i) for all $n \in \mathbb{N}$, $x_{n+1} = \sqrt{x_n y_n} > \sqrt{x_n x_n} = |x_n| = x_n$ and $(x_n)_{n=1}^{\infty}$ is strictly increasing.

(iii) Similarly for all $n \in \mathbb{N}$, $y_{n+1} = \frac{x_n+y_n}{2} < \frac{y_n+y_n}{2} = y_n$ and $(y_n)_{n=1}^{\infty}$ is strictly decreasing.

(iv) The first inequality was proven in (i) already. We now look at the second inequality. For $n = 1$, one simply needs to prove that $y_2 - x_2 < \frac{y_1-x_1}{2}$. But,

$$\begin{aligned}\tfrac{y_1-x_1}{2} - y_2 + x_2 &= \tfrac{y_1-x_1}{2} - \tfrac{y_1+x_1}{2} + \sqrt{x_1y_1} \\ &= \sqrt{x_1y_1} - x_1\end{aligned}$$

Since $x_1 < y_1$, $\sqrt{x_1y_1} - x_1 > 0$ and this yields that $y_2 - x_2 < \frac{y_1-x_1}{2}$.

Assume now that $y_{n+1} - x_{n+1} < \frac{y_1-x_1}{2^n}$. We need to show that $\frac{y_1-x_1}{2^{n+1}} - y_{n+2} - x_{n+2} > 0$. But,

$$\begin{aligned}\tfrac{y_1-x_1}{2^{n+1}} - y_{n+2} + x_{n+2} &= \tfrac{1}{2}\tfrac{y_1-x_1}{2^n} - \tfrac{y_{n+1}+x_{n+1}}{2} + \sqrt{x_{n+1}y_{n+1}} \\ &> \tfrac{1}{2}(y_{n+1} - x_{n+1}) - \tfrac{y_{n+1}+x_{n+1}}{2} + \sqrt{x_{n+1}y_{n+1}} \\ &= \sqrt{x_{n+1}y_{n+1}} - x_{n+1} > 0,\end{aligned}$$

and the induction is complete.

(v) By (ii), (iii) and the Monotone Convergence Theorem both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are convergent. By (iv) and the Squeeze Theorem $\lim_{n\to\infty}(y_n - x_n) = 0$ and thus $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

□

4 Subsequences

Exercise 15. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $a \in \mathbb{R}$. If the sequence $(x_n)_{n=1}^{\infty}$ does not converge to a, prove that there exists an $\varepsilon_0 > 0$ and a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, so that $|x_{n_k} - a| \geqslant \varepsilon_0$ for all $k \in \mathbb{N}$.

Hint. Negate the definition of convergence and construct the subsequence recursively. $\square$

Possible solution. Assume that $(x_n)_{n=1}^{\infty}$ does not converge to a. Then, there exists $\varepsilon_0 > 0$ so that

$$\text{for every } k \in \mathbb{N}, \text{ there exists } n_k \geqslant k \text{ with } |x_{n_k} - a| \geqslant \varepsilon_0. \tag{$*$}$$

We shall now construct the subsequence recursively. In particular, for $k = 1$, there exists $n_1 \in \mathbb{N}$ with $|x_{n_1} - a| \geqslant \varepsilon_0$. Assume now that there exist $n_1 < \cdots < n_k$ and $(x_{n_i})_{i=1}^{k}$ with $|x_{n_i} - a| \geqslant \varepsilon_0$ for $1 \leqslant i \leqslant k$. By $(*)$, there exists $N \geqslant n_k + 1$ with $|x_N - a| \geqslant \varepsilon_0$. Define $n_{k+1} = N$. Then, $n_{k+1} \geqslant n_k + 1 > n_k$, $|x_{n_{k+1}} - a| \geqslant \varepsilon_0$ and the recursive construction is complete. $\square$

Exercise 16. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $\ell \in \mathbb{R}$. Assume that for every subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, there exists a further subsequence $(z_n)_{n=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ that converges to ℓ. Prove that the original sequence $(x_n)_{n=1}^{\infty}$ converges to ℓ.

Hint. Argue by contradiction using Exercise 15. $\square$

Possible solution. If $(x_n)_{n=1}^{\infty}$ does not converge to ℓ, then by Exercise 15 there exist $\varepsilon_0 > 0$ and a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ so that

$$|x_{n_k} - \ell| \geqslant \varepsilon_0 \text{ for all } k \in \mathbb{N}. \tag{$**$}$$

By assumption, $(x_{n_k})_{k=1}^{\infty}$ has a further subsequence $(x_{n_{k_m}})_{m=1}^{\infty}$ that converges to ℓ and therefore there is $m_0 \in \mathbb{N}$ so that for all $m \geqslant m_0$, $|x_{n_{k_m}} - \ell| < \varepsilon_0$. As k_{m_0}, for instance, is still in $\mathbb{N}$, by $(**)$, $|x_{n_{k_{m_0}}} - \ell| \geqslant \varepsilon_0$. This contradiction completes the proof. $\square$

Exercise 17. For this exercise we will define a top point of a sequence $(x_n)_{n=1}^{\infty}$ as follows: we say that x_p is a top point of the sequence if for all $n \geq p$, $x_n \leq x_p$. Prove the monotone subsequence lemma using the notion of top point.

Hint. Consider the following three cases: the sequence has infinitely many top points, or finitely many top points, or no top points. $\square$

Possible solution. Assume first that $(x_n)_{n=1}^{\infty}$ has no top points. Let $k_1 = 1$. Since x_{k_1} is not a top point there exists $k_2 > k_1$ such that $x_{k_2} > x_{k_1}$. But x_{k_2} is not a top point either and there exists $k_3 > k_2 > k_1$ such that $x_{k_3} > x_{k_2} > x_{k_1}$. If we continue this process indefinitely we can construct recursively a subsequence $(x_{k_1})_{n=1}^{\infty}$ that is strictly increasing. Now, assume that a sequence $(x_n)_{n=1}^{\infty}$ has infinitely many top points then there exist $p_1 < p_2 < \cdots < p_k < \cdots$

such that for all $m \leq n$, $x_{p_m} \geq x_{p_n}$ and the subsequence $(x_{p_k})_{k=1}^{\infty}$ is decreasing. If $(x_n)_{n=1}^{\infty}$ has finitely many top points and let x_p the largest of those top points. Let $k_1 = p+1$, then x_{k_1} is not a top point and hence there exists $k_2 > k_1$ such that $x_{k_2} > x_{k_1}$. Since x_{k_2} is not a top point there exists $k_3 > k_2 > k_1$ such that $x_{k_3} > x_{k_2} > x_{k_1}$, and we can construct recursively a subsequence $(x_{k_n})_{n=1}^{\infty}$ that is (strictly) increasing. In all three cases, we were able to show the existence of a monotone subsequence. $\square$

Exercise 18. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let $t := \liminf_{n\to\infty} x_n$. Show that there exists a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_n = t$.

Hint. Construct the subsequence recursively using the approximation property for suprema and conclude with the Squeeze Theorem. $\square$

5 Constructing sequences

Exercise 19. Prove that for every real number x there exists a sequence of rational numbers $(q_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} q_n = x$.

Hint. Use the density of $\mathbb{Q}$ in $\mathbb{R}$ and the Squeeze Theorem. $\square$

Possible solution. Let x be a real number. Then by density of $\mathbb{Q}$ in $\mathbb{R}$, for every $n \in \mathbb{N}$ there exists $q_n \in \mathbb{Q}$ such that $x < q_n < x + \frac{1}{n}$. By the Squeeze Theorem $\lim_{n\to\infty} q_n = x$. $\square$

Exercise 20. Let X be a non-empty subset of $\mathbb{R}$ that is bounded above. Assume that $\sup(X) \notin X$. Prove that there exists a strictly increasing sequence $(x_n)_{n=1}^{\infty}$ of X so that $\lim_{n\to\infty} x_n = \sup(X)$.

Hint. Construct the sequence recursively using the approximation property for suprema. $\square$

Possible solution. Set $s = \sup(X)$. We will recursively choose for each $n \in \mathbb{N}$ a number $x_n \in X$, so that $x_1 < \cdots < x_n$ and $|x_n - s| < \frac{1}{n}$. This will yield the desired sequence. To do this rigorously we will use the Principle of Mathematical Induction. Let $P(n)$ be the statement: there exist $x_1 < \cdots < x_n$ elements in X such that $s - \frac{1}{n} < x_n \leqslant s$.

By the approximation property for suprema (for $\varepsilon = 1$), we may choose $x_1 \in X$ with $s - 1 < x_1 \leqslant s$ and $P(1)$ is true.

Assume now that there exist $x_1 < \cdots < x_n$ elements in X such that $s - \frac{1}{n} < x_n \leqslant s$. Since $x_n \leqslant s$ and $s \notin X$, we have $x_n < s$, i.e. $s - x_n > 0$. Set

$\varepsilon = \min\{\frac{1}{n+1}, s - x_n\}$, which is positive. By the approximation property of suprema we may choose $x_{n+1} \in X$ with $s - \varepsilon < x_{n+1} \leqslant s$. Since $s - \varepsilon < x_{n+1} \leqslant s < s + \varepsilon$, we conclude $|x_{n+1} - s| < \varepsilon \leqslant 1/(n+1)$. Furthermore, observe that $x_n = s - (s - x_n) \leqslant s - \varepsilon < x_{n+1}$, therefore x_{n+1} satisfies the desired properties. By the Principle of Mathematical Induction for all $n \in \mathbb{N}$ $P(n)$ is true, i.e. for every $n \in \mathbb{N}$ there exist $x_1 < \cdots < x_n$ elements in X such that $s - \frac{1}{n} < x_n \leqslant s$. The sequence $(x_n)_{n=1}^\infty$ is the desired sequence, since by the Squeeze Theorem $\lim_{n\to\infty} x_n = s$. □

Exercise 21. Let X be a non-empty subset of $\mathbb{R}$ that is bounded below. Assume that $\inf(X) \notin X$. Prove that there exists a strictly decreasing sequence $(x_n)_{n=1}^\infty$ of X so that $\lim_{n\to\infty} x_n = \inf(X)$.

Hint. Mimic the proof of Exercise 20. □

6 Cauchy Sequences

Exercise 22. Show that a Cauchy sequence is bounded.

Hint. The proof is similar to the proof of the fact that a convergent sequence is bounded. □

Possible solution. Assume that $(x_n)_{n=1}^\infty$ is Cauchy. Then for $\varepsilon = 1$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| \leq 1$. In particular for $m = N$ and by reverse triangle inequality $|x_n| \leq 1 + |x_N|$ for all $n \geq N$. Let $M := \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |x_N|\}$, then for all $n \in \mathbb{N}$, $|x_n| \leq M$ and $(x_n)_{n=1}^\infty$ is bounded. □

Exercise 23. Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be two Cauchy sequences such that $|y_n| \geq \alpha > 0$ for all $n \in \mathbb{N}$. Show that the sequence $(\frac{x_n}{y_n})_{n=1}^\infty$ is Cauchy.

Hint. Use the Triangle Inequality and ad-hoc algebraic manipulations. □

Possible solution. If follows from the triangle inequality and the assumptions that $|\frac{x_n}{y_n} - \frac{x_m}{y_m}| = |\frac{x_n y_m - y_n x_m}{y_n y_m}| = |\frac{(x_n - x_m)y_m - (y_n - y_m)x_m}{y_n y_m}| \leq |x_n - x_m|\frac{|y_n|}{\alpha^2} + |y_n - y_m|\frac{|x_m|}{\alpha^2}$. Since a Cauchy sequence is bounded (cf Exercise 22) there exists $M > 0$ such that for all $n \in \mathbb{N}$, $\max\{|y_n|, |x_n|\} \leq M$. Let $\varepsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n, m \geq N_1$, $|x_n - x_m| < \frac{\varepsilon\alpha^2}{2M}$ and for all $n, m \geq N_2$, $|y_n - y_m| < \frac{\varepsilon\alpha^2}{2M}$, and hence for $n, m \geq \max\{N_1, N_2\}$, $|\frac{x_n}{y_n} - \frac{x_m}{y_m}| \leq |x_n - x_m|\frac{|y_n|}{\alpha^2} + |y_n - y_m|\frac{|x_m|}{\alpha^2} < \frac{M}{\alpha^2}\frac{\varepsilon\alpha^2}{2M} + \frac{M}{\alpha^2}\frac{\varepsilon\alpha^2}{2M} < \varepsilon$. □

Exercise 24. Let $(x_n)_{n=1}^{\infty}$ be a sequence of integers, i.e. $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$.

(i) If $(x_n)_{n=1}^{\infty}$ is Cauchy, show that it is eventually constant (i.e. there exists $n_0 \in \mathbb{N}$ so that for all $n \geqslant n_0$ we have $x_n = x_{n_0}$).

(ii) If $(x_n)_{n=1}^{\infty}$ converges to some $\ell \in \mathbb{R}$, then $\ell \in \mathbb{Z}$.

Hint. For i) use the definition of Cauchy sequence for a well chosen ε and derive a contradiction if the sequence is not eventually constant. For ii) use i). □

Possible solution. (i) Fix $\varepsilon = 1/2$ (or any other number in $(0,1)$). As $(x_n)_{n=1}^{\infty}$ is Cauchy, there exists $N \in \mathbb{N}$, so that for all $m, n \in \mathbb{N}$ with $m \geqslant n \geqslant N$ we have $|x_n - x_m| < \varepsilon = 1/2$. In particular, for all $n \geqslant N$ we have $|x_n - x_N| < 1/2$. For $n \in \mathbb{N}$ with $n \geqslant N$, as $x_n - x_N$ is in $\mathbb{Z}$, it is either zero or $|x_n - x_N| \geqslant 1$. Since the second case is impossible, we conclude that $x_n = x_N$ for all $n \geq N$.

(ii) If $(x_n)_{n=1}^{\infty}$ converges to some $\ell \in \mathbb{R}$, it is Cauchy. By (i), there exists N so that $x_n = x_N$ for all $n \geqslant N$. This yields $\lim_{n\to\infty} x_n = x_N$ and hence $\ell = x_N \in \mathbb{Z}$. □

Exercise 25. Let $(x_n)_{n=1}^{\infty}$ be a sequence. Suppose that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m \geq n \geq N$, $|\sum_{k=n}^{m} x_k| < \varepsilon$. Prove that
$\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} x_k$ exists and is finite.

Hint. If you introduce a well chosen sequence it is a one line argument. □

Possible solution. Consider the sequence defined as $s_n = \sum_{k=1}^{n} x_k$, for $n \in \mathbb{N}$. Then if $m \geq n$, $|s_m - s_n| = |\sum_{k=n+1}^{m} x_k|$, and our assumption says that $(s_n)_{n=1}^{\infty}$ is Cauchy. Since every Cauchy sequence is convergent $(s_n)_{n=1}^{\infty}$ is convergent. □

Exercise 26. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq \frac{1}{3^n}$. Show that $(x_n)_{n=1}^{\infty}$ is convergent.

Hint. Show that $(x_n)_{n=1}^{\infty}$ is Cauchy. □

Possible solution. Let $m, n \in \mathbb{N}$. Without loss of generality we can assume that $m > n$. Then

$$\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - \cdots + x_{n+2} - x_{n+2} + x_{n+1} - x_n| \\
&= |\textstyle\sum_{i=n+1}^{m} x_i - \sum_{i=n}^{m-1} x_i| \\
&= |\textstyle\sum_{i=n}^{m-1} (x_{i+1} - x_i)| \\
&= \textstyle\sum_{i=n}^{m-1} |x_{i+1} - x_i| \text{ (by triangle inequality)} \\
&= \textstyle\sum_{i=n}^{m-1} \frac{1}{3^i} \text{ (by our assumption)}
\end{aligned}$$

It remains to show that $\sum_{i=n}^{m-1} \frac{1}{3^i}$ is small than whenever m, n are large enough. By induction one can show that

$$\sum_{i=0}^{k} \frac{1}{3^i} = \frac{3}{2}(1 - \frac{1}{3^{k+1}}).$$

Therefore,

$$\begin{aligned}
\sum_{i=n}^{m-1} \frac{1}{3^i} &= \frac{1}{3^n} \sum_{i=0}^{m-n-1} \frac{1}{3^i} \\
&= \frac{1}{3^n} \frac{3}{2}(1 - \frac{1}{3^{m-n}}) \\
&= \frac{3}{2}(\frac{1}{3^n} - \frac{1}{3^m}).
\end{aligned}$$

Now let $\varepsilon > 0$. Since $(\frac{1}{3^n})_{n=1}^{\infty}$ converges to 0, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|\frac{1}{3^n}| < \frac{\varepsilon}{3}$ and if $m > n \geq N$,

$$\begin{aligned}
|x_m - x_n| &\leq \textstyle\sum_{i=n}^{m-1} \frac{1}{3^i} \\
&\leq \frac{3}{2}(\frac{1}{3^n} - \frac{1}{3^m}) \\
&\leq \frac{3}{2}(\frac{1}{3^n} + \frac{1}{3^m}) \\
&< \frac{3}{2}(\frac{\varepsilon}{3} + \frac{\varepsilon}{3}) \\
&< \varepsilon,
\end{aligned}$$

and $(x_n)_{n=1}^{\infty}$ is Cauchy. $\square$