MATH 409, Summer 2019, Practice Problem Set 3

F. Baudier (Texas A&M University)

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1 Limits

Exercise 1. Use **only the definition of the limit of a function** to show if $a \in \mathbb{R}$ then $\lim\limits_{x\to a} x^2 = a^2$.

Exercise 2. Show that the function $f(x) = \sin(\frac{1}{x})$ does not have a limit at $x_0 = 0$.

Hint: Exhibit two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converging to 0 such that $(f(x_n))_{n=1}^{\infty}$ and $(f(y_n))_{n=1}^{\infty}$ do not have the same limit and invoke the sequential characterization of limits. $\square$

Exercise 3. Let $x_0 \in \mathbb{R}$ and assume that $f\colon (a,b)\setminus\{x_0\} \to (c,d)$ with $x_0 \in (a,b)$ and $g\colon (c,d) \to \mathbb{R}$. Show taht if f has a limit at x_0 and $\lim_{x\to x_0} f(x) := \ell \in (c,d)$ and if g is continuous at ℓ then $g\circ f$ has a limit at x_0 and $\lim_{x\to x_0} g\circ f(x) := g(\lim_{x\to x_0} f(x))$.

Hint. $\square$

Exercise 4. Let $x_0 \in (a,b)$ and assume that $f\colon (a,b)\setminus\{x_0\} \to \mathbb{R}$. Assume that f has a limit at x_0. If $\lim_{x\to x_0} f(x) \neq 0$ show that:

1. there exist $\alpha > 0$ an $\delta > 0$ such that if $|x - x_0| < \delta$ and $x \neq x_0$ then $|f(x)| > \alpha$,
2. $\lim\limits_{x\to x_0} \dfrac{1}{f(x)} = \dfrac{1}{\lim\limits_{x\to x_0} f(x)}$ without using the sequential characterization of limits.

Hint. For 1. find inspiration on the analogous result for sequences and for 2. use 1.. □

Exercise 5. Prove the comparison theorem for functions without using the sequential characterization of limits.

Hint. Find inspiration in the proof of the comparison theorem for sequences. □

Exercise 6. Prove the squeeze theorem for functions without using the sequential characterization of limits.

Hint. Find inspiration in the proof of the squeeze theorem for sequences. □

Exercise 7. Show that $\lim_{\theta \to 0} \dfrac{\sin(\theta)}{\theta} = 1$.

Hint. Show that $\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}$ and use the squeeze theorem. □

Exercise 8. Prove the Squeeze Theorem for functions.

Hint: Either mimic the proof of the Squeeze Theorem for sequences of use the sequential characterization of limits together with the Squeeze Theorem for sequences. □

Exercise 9. Let $f \colon (a, b) \to \mathbb{R}$, $x_0 \in (a, b)$ and $\ell \in \mathbb{R}$. Show that,

$\lim_{x \to x_0} f(x) = \ell$ if and only if $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \ell$.

2 Continuity

Exercise 10. Let $x_0 \in \mathbb{R}$ and assume that $f \colon (a, b) \to (c, d)$ with $x_0 \in (a, b)$ and $g \colon (c, d) \to \mathbb{R}$. Show that if f is continuous at x_0 and if g is continuous at $f(x_0)$ then $g \circ f$ is continuous at x_0.

Hint. You can exploit the definitions. □

We define the notion of open set.

Definition 1 (Open set)**.** A subset U of $\mathbb{R}$ is open if for every $x \in U$ there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq U$.

Exercise 11. Let $a < b$. Show that (a, b) is open.

Hint. Exploit the definitions. $\square$

Exercise 12 (Characterization of continuity in terms of open sets). Let $f\colon \mathbb{R} \to \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$ if and only if for every open subset U of $\mathbb{R}$, $f^{-1}(U)$ is open.

Hint. Use the $\varepsilon - \delta$ definition of continuity and the previous exercise. $\square$

We now define the notion of compact set.

Definition 2 (Compact set)**.** A subset K of $\mathbb{R}$ is compact if every open cover of K admits a finite open subcover, i.e. if $K \subseteq \bigcup_{i\in I} U_i$ where U_i is open for all $i \in I$ then there exists $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$ such that $K \subseteq \bigcup_{k=1}^{n} U_{i_k}$.

Exercise 13 (The continuous image of a compact set is compact). Let $f\colon \mathbb{R} \to \mathbb{R}$. be continuous. Prove that if K is compact, then $f(K)$ is compact.

Hint. Use the previous exercise. $\square$

Exercise 14. [Converse of the Intermediate Value Theorem for increasing functions] Let $f\colon [a,b] \to \mathbb{R}$ be an increasing function such that $f(a) < f(b)$ and whenever $f(a) < y_0 < f(b)$ there exists $x_0 \in (a,b)$ such that $f(x_0) = y_0$. Show that f is continuous on $[a,b]$.

Hint. Use the definition of continuity and the monotonicity of the function. $\square$

3 Applications of the Intermediate Value Theorem

Exercise 15. A function f is said to have a fixed point in $[a,b]$ is there exist $c \in [a,b]$ such that $f(c) = c$. Let $f\colon [a,b] \to [a,b]$ be a continuous function. Show that f has a fixed point in $[a,b]$.

Hint. Consider the function $g : [a,b] \to \mathbb{R}$ with $g(x) = x - f(x)$. $\square$

Exercise 16. Let I be an interval and let $f: I \to \mathbb{R}$. Assume f is continuous on I and f is injective. Show that either f is strictly increasing or strictly decreasing on I.

Hint. Try a proof by contradiction. The key point is a correct negation of the statement "f is strictly increasing or strictly decreasing on I". □

Exercise 17. Let $f: [a, b] \to \mathbb{R}$ be continuous and injective on $[a, b]$, then $f([a, b])$ is a closed bounded interval and the inverse of f onto its image, $f^{-1}: f([a, b]) \to [a, b]$, is continuous.

Hint: Show that f^{-1} is strictly monotone using Exercise 16 and then use Exercise 14. □

4 Uniform Continuity

Exercise 18. Show that the function $f(x) = x^2$ is uniformly continuous on $(-1, 1)$ but not on $\mathbb{R}$.

Exercise 19. Show that the function $f(x) = \sin(\frac{1}{x})$ is not uniformly continuous on $(0, 2)$.

Exercise 20. A function $f: [a, b] \to \mathbb{R}$ is Lipschitz if there exists $C > 0$ such that $|f(x_1) - f(x_2)| \leq C|x_1 - x_2|$ for all $x_1, x_2 \in [a, b]$. Show that f is uniformly continuous on $[a, b]$.

Exercise 21. Let $f: (a, b) \to \mathbb{R}$ be uniformly continuous and $(x_n)_{n=1}^{\infty}$ a Cauchy sequence of elements in (a, b). Show that $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence.

Hint. It follows from the definitions. □

Exercise 22. Let $f: (a, b) \to \mathbb{R}$. Show that f is uniformly continuous on (a, b) if and only if there is a continuous function $g: [a, b] \to \mathbb{R}$ which extends f, i.e. g satisfies $g(x) = f(x)$ for all $x \in (a, b)$.

Hint. Use the previous exercise. □

5 Applications of the Extreme Value Theorem

Exercise 23. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and let $\varepsilon > 0$. Prove that there exists $n \in \mathbb{N}$ so that for $k = 1, \dots, n$ we have

$$\sup\left\{f(x) : \frac{k-1}{n} \leqslant x \leqslant \frac{k}{n}\right\} - \inf\left\{f(x) : \frac{k-1}{n} \leqslant x \leqslant \frac{k}{n}\right\} < \varepsilon.$$

Hint. Use the Extreme Value Value Theorem and uniform continuity. □

For the next exercise we recall the following definition.

Definition 3. Let $f : \mathbb{R} \to \mathbb{R}$. We say that f diverges to $+\infty$ when x tends to $+\infty$ if for all $M > 0$ there exists $x_0 \in \mathbb{R}$ such that for all $x > x_0$, $f(x) > M$. And we say that f diverges to $+\infty$ when x tends to $-\infty$ if for all $M > 0$ there exists $x_0 \in \mathbb{R}$ such that for all $x < x_0$, $f(x) > M$.

Exercise 24. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty.$$

Prove that there exists $x_m \in \mathbb{R}$ such that $f(x_m) = \inf\{f(x) : x \in \mathbb{R}\}$.

Hint. Use the Extreme Value Value Theorem. □

6 Pathological functions

We start with a preliminary result that will be needed in a later exercise.

Exercise 25 (Density of the irrationals in the reals).

1. Prove that $\sqrt{2}$ is irrational.
2. Prove that the irrational are dense in $\mathbb{R}$, i.e. for every real number $x < y$ there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ so that $x < \alpha < y$.

Hint. For (2) use (1). □

Exercise 26 (A function discontinuous everywhere). The Dirichlet function is defined on $\mathbb{R}$ by

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that every point $x \in \mathbb{R}$ is a point of discontinuity of $\chi_{\mathbb{Q}}$.

Hint. Use the density of the irrationals to prove that every rational is a point of discontinuity. Use the density of the rationals to prove that every irrational is a point of discontinuity. $\square$

Exercise 27 (A function discontinuous at every rational point but continuous at every irrational in $(0,1)$). Let g be defined on $\mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ (in reduced form).} \end{cases}$$

Prove that g is discontinuous at every rational point but continuous at every irrational in $(0,1)$.

Hint. Use the density of the irrationals to prove discontinuity at every rational. To prove continuity at every irrational in $(0,1)$ consider the set $F = \{\frac{p}{q} : 0 < \frac{p}{q} < 1 \text{ and } 2 \leqslant q \leqslant N\}$ for some well chosen N. $\square$

Exercise 28 (A function discontinuous at every irrational point but continuous at every rational point). Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ be an enumeration of the set of rational numbers (i.e. for each $q \in \mathbb{Q}$ there is exactly one $n \in \mathbb{N}$ such that $q = q_n$). Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the rule

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{n} & \text{if } x \in \mathbb{Q} \text{ and } x = q_n. \end{cases}$$

Prove that f is continuous at x if and only if $x \in \mathbb{R} \setminus \mathbb{Q}$.

Hint. Show that for all $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ we have $\lim_{x \to x_0} f(x) = 0$. You might want to consider the set $F_N = \{q_n : 1 \leqslant n \leqslant N\}$ for some well chosen N. $\square$