MATH 409, Summer 2019, Practice Problem Set 3

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1 Limits

Exercise 1. Use **only the definition of the limit of a function** to show if $a \in \mathbb{R}$ then $\lim\limits_{x\to a} x^2 = a^2$.

Solution. Fix $\varepsilon > 0$ and set $\delta = \min\left\{\frac{\varepsilon}{2|a|+1}, 1\right\}$. Then, if $x \in \mathbb{R}$ with $0 < |x-a| < \delta$, we have

$$\begin{aligned}|x^2 - a^2| &= |x-a| \cdot |x+a| < \delta|(x-a)+2a| \leqslant \delta(\delta + 2|a|) \\ &\leqslant \delta(1+2|a|) \leqslant \frac{\varepsilon}{2|a|+1}(1+2|a|) = \varepsilon.\end{aligned}$$

□

Exercise 2. Show that the function $f(x) = \sin(\frac{1}{x})$ does not have a limit at $x_0 = 0$.

Hint: Exhibit two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converging to 0 such that $(f(x_n))_{n=1}^{\infty}$ and $(f(y_n))_{n=1}^{\infty}$ do not have the same limit and invoke the sequential characterization of limits. □

Possible solution. For $n \geq 1$ let $x_n = \frac{1}{n\pi}$ and $y_n = \frac{1}{\frac{\pi}{2}+2n\pi}$, then both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converge to 0 but $\lim_{n\to\infty} f(x_n) = 0$ while $\lim_{n\to\infty} f(y_n) = 1$. By the sequential characterization of limits f does not have a limit at 0. □

Exercise 3. Let $x_0 \in \mathbb{R}$ and assume that $f\colon (a,b)\setminus\{x_0\} \to (c,d)$ with $x_0 \in (a,b)$ and $g\colon (c,d) \to \mathbb{R}$. Show that if f has a limit at x_0 and $\lim_{x\to x_0} f(x) := \ell \in (c,d)$ and if g is continuous at ℓ then $g \circ f$ has a limit at x_0 and $\lim_{x\to x_0} g \circ f(x) := g(\lim_{x\to x_0} f(x))$.

Hint. □

Solution. □

Exercise 4. Let $x_0 \in (a,b)$ and assume that $f\colon (a,b) \setminus \{x_0\} \to \mathbb{R}$. Assume that f has a limit at x_0. If $\lim_{x\to x_0} f(x) \neq 0$ show that:

1. there exist $\alpha > 0$ an $\delta > 0$ such that if $|x - x_0| < \delta$ and $x \neq x_0$ then $|f(x)| > \alpha$,
2. $\displaystyle\lim_{x\to x_0} \frac{1}{f(x)} = \frac{1}{\lim_{x\to x_0} f(x)}$ without using the sequential characterization of limits.

Hint. For 1. find inspiration on the analogous result for sequences and for 2. use 1.. □

Solution.

1. Assume that $\lim_{x\to x_0} f(x) = \ell \neq 0$. Let $\varepsilon_0 = \frac{|\ell|}{2} > 0$, then there exists $\delta > 0$ such that if $|x - x_0| < \delta$ and $x \neq x_0$ then $|f(x) - \ell| < \varepsilon_0$ and by reverse triangle inequality $|f(x)| > -\varepsilon_0 + |\ell| = \frac{|\ell|}{2} > 0$. So $\alpha = \frac{|\ell|}{2}$ will do.
2. Assume $\lim_{x\to x_0} f(x) = \ell \neq 0$, then by (1) there exist $\alpha > 0$ an $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$ and $x \neq x_0$ then $|f(x)| > \alpha$. Let $\varepsilon > 0$, then there exists $\delta_2 > 0$ such that if $|x - x_0| < \delta_2$ and $x \neq x_0$, then $|f(x) - \ell| < \varepsilon\alpha|\ell|$. For $\delta = \min\{\delta_1, \delta_2\}$, if $|x - x_0| < \delta$ and $x \neq x_0$, then $|\frac{1}{f(x)} - \frac{1}{\ell}| = |\frac{\ell - f(x)}{f(x)\ell}| < \frac{|f(x)-\ell|}{\alpha|\ell|} < \varepsilon$.

□

Exercise 5. Prove the comparison theorem for functions without using the sequential characterization of limits.

Hint. Find inspiration in the proof of the comparison theorem for sequences. □

Solution. □

Exercise 6. Prove the squeeze theorem for functions without using the sequential characterization of limits.

Hint. Find inspiration in the proof of the squeeze theorem for sequences. □

Solution. □

Exercise 7. Show that $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1$.

Hint. Show that $\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}$ and use the squeeze theorem. □

Solution. By comparing areas of two triangles with ad-hoc side lengths and the area of a region subtended by and arc of angle θ in the trigonometric circle one can easily get the desired inequalities. We conclude by the squeeze theorem since $\lim_{\theta\to 0}\cos(\theta) = 1$. □

Exercise 8. Prove the Squeeze Theorem for functions.

Hint: Either mimic the proof of the Squeeze Theorem for sequences of use the sequential characterization of limits together with the Squeeze Theorem for sequences. □

Exercise 9. Let $f\colon (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$ and $\ell \in \mathbb{R}$. Show that,

$\lim_{x\to x_0} f(x) = \ell$ if and only if $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \ell$.

Hint. One implication is immediate the other one follows from the definitions. □

Solution. If $\lim_{x\to x_0} f(x) = \ell$ then by definition of a two-sided limit we immediately have that $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \ell$. For the converse, assume that $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \ell$. Let $\varepsilon > 0$, then there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x \in (a,b)$ such that if $x_0 < x < x_0+\delta_1$, then $|f(x)-\ell| < \varepsilon$ and if $x_0 - \delta_2 < x < x_0$, then $|f(x) - \ell| < \varepsilon$. Now take $\delta = \min\{\delta_1, \delta_2\} > 0$. If $|x - x_0| < \delta$ and $x \neq x_0$, then $|f(x) - \ell| < \varepsilon$. □

2 Continuity

Exercise 10. Let $x_0 \in \mathbb{R}$ and assume that $f\colon (a,b) \to (c,d)$ with $x_0 \in (a,b)$ and $g\colon (c,d) \to \mathbb{R}$. Show that if f is continuous at x_0 and if g is continuous at $f(x_0)$ then $g \circ f$ is continuous at x_0.

Hint. You can exploit the definitions. □

Solution. □

We define the notion of open set.

Definition 1 (Open set)**.** A subset U of $\mathbb{R}$ is open if for every $x \in U$ there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$.

Exercise 11. Let $a < b$. Show that (a, b) is open.

Hint. Exploit the definitions. □

Exercise 12 (Characterization of continuity in terms of open sets). Let $f: \mathbb{R} \to \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$ if and only if for every open subset U of $\mathbb{R}$, $f^{-1}(U)$ is open.

Hint. Use the $\varepsilon - \delta$ definition of continuity and the previous exercise. □

Solution. Assume that f is continuous and let U be an open subset of $\mathbb{R}$. By definition $f^{-1}(U) = \{x \in \mathbb{R}: f(x) \in U\}$. If $f^{-1}(U) = \emptyset$ then $f^{-1}(U)$ is trivially open. Otherwise let $x \in f^{-1}(U)$, then $f(x) \in U$ and there exists $\varepsilon > 0$ such that $(f(x) - \varepsilon, f(x) + \varepsilon) \subseteq U$. Thus, $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon)) \subseteq f^{-1}(U)$, but $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon)) = \{y \in \mathbb{R}: f(x) - \varepsilon < f(y) < f(x) + \varepsilon\} = \{y \in \mathbb{R}: |f(y) - f(x)| < \varepsilon\}$. By continuity of f at x there exists $\delta > 0$ such that if $|y - x| < \delta$ then $|f(y) - f(x)| < \varepsilon$, which translated in terms of sets means that $\{y \in \mathbb{R}: |y - x| < \delta\} \subset \{y \in \mathbb{R}: |f(y) - f(x)| < \varepsilon\}$. In other words, for every $x \in f^{-1}(U)$ there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset f^{-1}(U)$ and $f^{-1}(U)$ is open.

For the converse, assume that for every open subset U of $\mathbb{R}$, $f^{-1}(U)$ is open. Let $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ and consider the set $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ which is open by (1). Note that $x_0 \in f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ since $f(x_0) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and hence there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$, which means that for every $x \in (x_0 - \delta, x_0 + \delta)$ one has $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, and f is continuous at x_0. Since x_0 was fixed but arbitrary f is continuous on $\mathbb{R}$. □

We now define the notion of compact set.

Definition 2 (Compact set)**.** A subset K of $\mathbb{R}$ is compact if every open cover of K admits a finite open subcover, i.e. if $K \subseteq \bigcup_{i \in I} U_i$ where U_i is open for all $i \in I$ then there exists $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$ such that $K \subseteq \bigcup_{k=1}^{n} U_{i_k}$.

Exercise 13 (The continuous image of a compact set is compact). Let $f: \mathbb{R} \to \mathbb{R}$. be continuous. Prove that if K is compact, then $f(K)$ is compact.

Hint. Use the previous exercise. □

Solution. Assume that K is compact and let $(U_i)_{i\in I}$ be an open covering of $f(K)$, i.e. $f(K) \subseteq \bigcup_{i\in I} U_i$ where U_i is open for all $i \in I$. Then, $K \subseteq f^{-1}(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f^{-1}(U_i)$, and $\bigcup_{i\in I} f^{-1}(U_i)$ is an open covering of K. Indeed by (1) $f^{-1}(U_i)$ is open since f is continuous. By compactness of K there exists $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$ such that $K \subseteq \bigcup_{k=1}^{n} f^{-1}(U_{i_k})$ and $f(K) \subseteq \bigcup_{k=1}^{n} U_{i_k}$, and $f(K)$ is compact. □

Exercise 14. [Converse of the Intermediate Value Theorem for increasing functions] Let $f\colon [a,b] \to \mathbb{R}$ be an increasing function such that $f(a) < f(b)$ and whenever $f(a) < y_0 < f(b)$ there exists $x_0 \in (a,b)$ such that $f(x_0) = y_0$. Show that f is continuous on $[a,b]$.

Hint. Use the definition of continuity and the monotonicity of the function. □

Solution. Assume that f is increasing. Let $x_0 \in (a,b)$ and $\varepsilon > 0$. If $f(a) \geq f(x_0) - \varepsilon$, let $c_1 = a$ and notice that if $c_1 < x < x_0$, then $0 \leq f(x_0) - f(x) \leq f(x_0) - f(a) \leq \varepsilon$. Otherwise, let $y_0 = f(x_0) - \varepsilon$ and thus $f(a) < y_0 = f(x_0) - \varepsilon < f(x_0) \leq f(b)$. Therefore, by assumption, there exists $c_1 \in (a,b)$ such that $f(c_1) = y_0$. Since $f(c_1) = \alpha = f(x_0) - \varepsilon < f(x_0)$, it must be the case that $c_1 < x_0$ as f is increasing. Furthermore, if $c_1 < x < x_0$, then $0 \leq f(x_0) - f(x) \leq f(x_0) - f(c_1) = f(x_0) - y_0 = f(x_0) - (f(x_0) - \varepsilon) = \varepsilon$. Hence in either case, there exists $c_1 \in [a, x_0)$ such that $|f(x) - f(x_0)| \leq \varepsilon$ for all $x \in (c_1, x_0)$.

Now if $f(b) \leq f(x_0) + \varepsilon$, let $c_2 = b$ and notice that if $x_0 < x < c_2$ then $0 \leq f(x) - f(x_0) \leq f(b) - f(x_0) \leq \varepsilon$. Otherwise, $f(b) > f(x_0) + \varepsilon$, and let $y_0 = f(x_0) + \varepsilon$. Therefore, by assumptions, there exists $c_2 \in (a,b)$ such that $f(c_2) = y_0$. Since $f(c_2) = y_0 = f(x_0) + \varepsilon > f(x_0)$, it must be the case that $c_2 > x_0$ as f is increasing. Furthermore, if $c_2 > x > x_0$, then $0 \leq f(x) - f(x_0) \leq f(c_2) - f(x_0) = y_0 - f(x_0) = f(x_0) + \varepsilon - f(x_0) = \varepsilon$. Hence in either case, there exists $c_2 \in (x_0, b]$ such that $|f(x) - f(x_0)| \leq \varepsilon$ for all $x \in (x_0, c_2)$. Therefore, if we let $\delta = \min\{x_0 - c_1, c_2 - x_0\} > 0$ it follows that for all $x \in (x_0 - \delta, x_0 + \delta)$, $|f(x) - f(x_0)| \leq \varepsilon$, and hence f is continuous at x_0. Continuity at a or b can be shown similarly. □

3 Applications of the Intermediate Value Theorem

Exercise 15. A function f is said to have a fixed point in $[a,b]$ if there exists $c \in [a,b]$ such that $f(c) = c$. Let $f\colon [a,b] \to [a,b]$ be a continuous function. Show that f has a fixed point in $[a,b]$.

Hint. Consider the function $g : [a,b] \to \mathbb{R}$ with $g(x) = x - f(x)$. □

Possible solution. Define $g : [a, b] \to \mathbb{R}$ with $g(x) = x - f(x)$. Then g is continuous as the difference of continuous functions. As the image of $[a, b]$ under f is contained in $[a, b]$, we deduce $f(a) \geqslant a$ and therefore $g(a) = a - f(a) \leqslant 0$. Similarly, we obtain $g(b) = b - f(b) \geqslant 0$.

If it so happens that $g(a) = 0$, then $f(a) = a$ and a is the desired number. Similarly, if $g(b) = 0$ then $f(b) = b$ and b is the desired number. If neither of the above happens to be true, then $g(a) < 0 < g(b)$. Applying the intermediate value theorem to g, we conclude that there if $c \in (a, b)$ so that $g(c) = 0$, i.e. $f(c) = c$. □

Exercise 16. Let I be an interval and let $f : I \to \mathbb{R}$. Assume f is continuous on I and f is injective. Show that either f is strictly increasing or strictly decreasing on I.

Hint. Try a proof by contradiction. The key point is a correct negation of the statement "f is strictly increasing or strictly decreasing on I". □

Solution. Assume that f is continuous and injective on I. Assume by contradiction that f is neither strictly increasing nor strictly decreasing then there exist $x_1 < x_2 < x_3$ in I such that $f(x_1) \leq f(x_2)$ and $f(x_3) \leq f(x_2)$ (or $f(x_1) \geq f(x_2)$ and $f(x_3) \geq f(x_2)$) (why?). Since the proof for the latter case is similar to the proof of the former case we only treat the case where $f(x_1) \leq f(x_2)$ and $f(x_3) \leq f(x_2)$. Since f is injective $f(x_1) < f(x_2)$ and $f(x_3) < f(x_2)$. Let α such that $f(x_1) < \alpha < f(x_2)$ and $f(x_3) < \alpha < f(x_2)$ (why such an α exists?). Since f is continuous on $[x_1, x_2]$, the IVT implies that there exists $c \in (x_1, x_2)$ such that $f(c) = \alpha$. Similarly, since f is continuous on $[x_2, x_3]$, the IVT implies that there exists $d \in (x_2, x_3)$ such that $f(d) = \alpha$ and thus $f(c) = f(d)$ for some $x_1 < c < x_2 < d < x_3$ which contradicts the injectivity of f. Therefore f is either strictly increasing or strictly decreasing. □

Exercise 17. Let $f : [a, b] \to \mathbb{R}$ be continuous and injective on $[a, b]$, then $f([a, b])$ is a closed bounded interval and the inverse of f onto its image, $f^{-1} : f([a, b]) \to [a, b]$, is continuous.

Hint: Show that f^{-1} is strictly monotone using Exercise 16 and then use Exercise 14. □

Proof. Note that f is either strictly increasing or strictly decreasing by Exercise 16. We will assume that f is strictly increasing as the proof in the case f is strictly decreasing will follow by similar arguments (or by considering $g = -f$). Since f is strictly increasing, we obtain $f(a) < f(b)$. Since f is continuous, we obtain by the Intermediate Value Theorem that $f([a, b]) = [f(a), f(b)]$. We claim that f^{-1} is strictly increasing. To see this, suppose $y_1, y_2 \in f([a, b])$ are such that $y_1 < y_2$. Choose $x_1, x_2 \in [a,b]$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $f(x_1) < f(x_2)$, it must be the case that $x_1 < x_2$ as f was strictly increasing. Hence $f^{-1}(y_1) = f^{-1}(f(x_1)) = x_1 < x_2 = f^{-1}(f(x_2)) = f^{-1}(y_2)$. Hence f^{-1} is strictly increasing. Therefore, $f^{-1}\colon [f(a), f(b)] \to [a,b]$ is a strictly increasing function such that $f^{-1}([f(a), f(b)]) = [a,b]$. Therefore f^{-1} is continuous by the converse of the Intermediate Value Theorem for monotone functions as f^{-1} satisfies the conclusions of the Intermediate Value Theorem. □

4 Uniform Continuity

Exercise 18. Show that the function $f(x) = x^2$ is uniformly continuous on $(-1,1)$ but not on $\mathbb{R}$.

Solution. Let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{2}$. If $x, y \in (-1,1)$ are such that $|x-y| < \delta$, then $|x^2 - y^2| = |x-y||x+y| \leq |x-y|(|x|+|y|) \leq 2|x-y| < 2\delta < \varepsilon$ and the function $x \mapsto x^2$ is uniformly continuous on $(-1,1)$. Now for every $n \in \mathbb{N}$, let $x_n = n$ and $y_n = n - \frac{1}{n}$. Then, $|x_n - y_n| = |n - (n - \frac{1}{n})| = \frac{1}{n}$ but $|x_n^2 - y_n^2| = |n^2 - (n - \frac{1}{n})^2| = |n^2 - (n^2 - 2 + \frac{1}{n^2})| = 2 - \frac{1}{n^2} \geq 1$ and f is not uniformly continuous on $\mathbb{R}$. □

Exercise 19. Show that the function $f(x) = \sin(\frac{1}{x})$ is not uniformly continuous on $(0,2)$.

Solution. Let $x_n = \frac{1}{2\pi(n+1)}$ and $y_n = \frac{1}{\frac{\pi}{2}+2\pi n}$, for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, $x_n, y_n \in (0,2)$, and $|x_n - y_n| = \frac{1}{\frac{\pi}{2}+2\pi n} - \frac{1}{2\pi(n+1)} = \frac{4}{2\pi(1+4n)} - \frac{1}{2\pi(n+1)} \leq \frac{1}{2\pi n} - \frac{1}{4\pi n} = \frac{1}{4\pi n}$, but $|\sin(\frac{1}{x_n}) - \sin(\frac{1}{y_n})| = |\sin(2\pi(n+1)) - \sin(\frac{\pi}{2} + 2\pi n)| = |0-1| = 1$. Therefore, f is not uniformly continuous on $(0,2)$ □

Exercise 20. A function $f\colon [a,b] \to \mathbb{R}$ is Lipschitz if there exists $C > 0$ such that $|f(x_1) - f(x_2)| \leq C|x_1 - x_2|$ for all $x_1, x_2 \in [a,b]$. Show that f is uniformly continuous on $[a,b]$.

Hint. It is straightforward from the definition. □

Solution. Let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{C}$, then if $x, y \in [a,b]$ are such that $|x-y| < \delta$ then $|f(x) - f(y)| \leq C|x-y| < C\frac{\varepsilon}{C} = \varepsilon$. □

Exercise 21. Let $f\colon (a,b) \to \mathbb{R}$ be uniformly continuous and $(x_n)_{n=1}^{\infty}$ a Cauchy sequence of elements in (a,b). Show that $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence.

Hint. It follows from the definitions. □

Solution. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence of element in (a,b) and $\varepsilon > 0$. By uniform continuity of f, there exists $\delta > 0$ such that for all $x, y \in (a,b)$, if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Since $(x_n)_{n=1}^{\infty}$ is Cauchy there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < \delta$ and thus $|f(x_n) - f(x_m)| < \varepsilon$. □

Exercise 22. Let $f\colon (a,b) \to \mathbb{R}$. Show that f is uniformly continuous on (a,b) if and only if there is a continuous function $g\colon [a,b] \to \mathbb{R}$ which extends f, i.e. g satisfies $g(x) = f(x)$ for all $x \in (a,b)$.

Hint. Use the previous exercise. □

Solution. Assume that there is a continuous function $g\colon [a,b] \to \mathbb{R}$ which extends f. Then g is uniformly continuous on $[a,b]$ and thus on (a,b) and f being the restriction of g on (a,b) it is also uniformly continuous on (a,b). Assume now that f is uniformly continuous on (a,b). Define $g\colon (a,b) \to \mathbb{R}$ by $g(x) = f(x)$. The function g is clearly continuous on (a,b). It remains to show that $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist and are finite and set $g(a) = \lim_{x\to a^+} f(x)$ and $g(b) = \lim_{x\to b^-} f(x)$ to complete the proof. Let $(z_n)_{n=1}^{\infty}$ be a sequence in (a,b) that is convergent to a. Then $(x_n)_{n=1}^{\infty}$ is Cauchy and by the previous exercise $(f(z_n))_{n=1}^{\infty}$ is also Cauchy. Since every Cauchy sequence of real numbers is convergent $(f(z_n))_{n=1}^{\infty}$ converges to some real number ℓ_z. At this point we still need to justify that the limit does not depend on the sequence. Let $(z_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequence in (a,b) that converge to a and such that $(f(z_n))_{n=1}^{\infty}$ converges to some real number ℓ_z and $(f(y_n))_{n=1}^{\infty}$ converges to some real number ℓ_y. Let $\varepsilon > 0$, and note that $|\ell_z - \ell_y| \leq |\ell_z - f(z_n)| + |f(z_n) - f(y_n)| + |f(y_n) - \ell_y|$. But there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|\ell_z - f(z_n)| < \frac{\varepsilon}{3}$, $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|\ell_y - f(y_n)| < \frac{\varepsilon}{3}$. There is also $N_3 \in \mathbb{N}$ such that for all $n \geq N_3$, $|f(z_n) - f(y_n)| < \frac{\varepsilon}{3}$. Indeed, since f is uniformly continuous on (a,b) there exists $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{3}$. Let $K_1 \geq 1$ such that for all $n \geq K_1$, $|x_n - a| < \frac{\delta}{2}$ and $K_2 \geq 1$ such that $|y_n - a| < \frac{\delta}{2}$ then for $n \geq \max\{K_1, K_2\}$ one has $|x_n - y_n| < \delta$ and thus $|f(x_n) - f(y_n)| < \frac{\varepsilon}{3}$. So if $N_3 = \max\{K_1, K_2\}$ then for all $n \geq N_3$, $|f(x_n) - f(y_n)| < \frac{\varepsilon}{3}$. Therefore, if $n \geq \max\{N_1, N_2, N_3\}$, $|\ell_z - \ell_y| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. We just proved that for all $\varepsilon > 0$, $|\ell_z - \ell_y| < \varepsilon$ which implies that $\ell_z = \ell_y$. By sequential characterization of limits, $\lim_{x\to a^+} f(x)$ exists and is finite and we set $g(a) = \lim_{x\to a^+} f(x)$. The case of b is identical. □

5 Applications of the Extreme Value Theorem

Exercise 23. Let $f : [0,1] \to \mathbb{R}$ be a continuous function and let $\varepsilon > 0$. Prove that there exists $n \in \mathbb{N}$ so that for $k = 1, \ldots, n$ we have

$$\sup\left\{f(x) : \frac{k-1}{n} \leqslant x \leqslant \frac{k}{n}\right\} - \inf\left\{f(x) : \frac{k-1}{n} \leqslant x \leqslant \frac{k}{n}\right\} < \varepsilon.$$

Hint. Use the Extreme Value Value Theorem and uniform continuity. □

Solution. Since f is continuous on $[0,1]$, it is uniformly continuous on $[0,1]$. Hence, there exists $\delta > 0$, that for all x, $y \in [0,1]$ with $|x - y| < \delta$, one has $|f(x) - f(y)| < \varepsilon$. By the Archimedean Property, there exists $n \in \mathbb{N}$, such that $1/n \leqslant \delta$. We will show that this n satisfies the desired property.

Let $1 \leqslant k \leqslant n$. By the Extreme Value Theorem, there exist x_0, y_0 in $[(k-1)/n, k/n]$, so that $f(x_0) = \sup\{f(x) : \frac{k-1}{n} \leqslant x \leqslant \frac{k}{n}\}$ and $f(y_0) = \inf\{f(x) : \frac{k-1}{n} \leqslant x \leqslant \frac{k}{n}\}$. As $x_0, y_0 \in [(k-1)/n, k/n]$, we have $|x_0 - y_0| < 1/n \leqslant \delta$ and therefore $|f(x_0) - f(y_0)| < \varepsilon$. In conclusion,

$$\begin{aligned}&\sup\left\{f(x) : \frac{k-1}{n} \leqslant x \leqslant \frac{k}{n}\right\} - \inf\left\{f(x) : \frac{k-1}{n} \leqslant x \leqslant \frac{k}{n}\right\}\\ &= f(x_0) - f(y_0) \leqslant |f(x_0) - f(y_0)| < \varepsilon.\end{aligned}$$

□

For the next exercise we recall the following definition.

Definition 3. Let $f : \mathbb{R} \to \mathbb{R}$. We say that f diverges to $+\infty$ when x tends to $+\infty$ if for all $M > 0$ there exists $x_0 \in \mathbb{R}$ such that for all $x > x_0$, $f(x) > M$. And we say that f diverges to $+\infty$ when x tends to $-\infty$ if for all $M > 0$ there exists $x_0 \in \mathbb{R}$ such that for all $x < x_0$, $f(x) > M$.

Exercise 24. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty.$$

Prove that there exists $x_m \in \mathbb{R}$ such that $f(x_m) = \inf\{f(x) : x \in \mathbb{R}\}$.

Hint. Use the Extreme Value Value Theorem. □

Solution. Let $y_0 = f(0)$. Since $\lim_{x\to-\infty} f(x) = +\infty$, there exists $x_1 \in \mathbb{R}$ such that for all $x < x_1$, $f(x) > y_0$ and since $\lim_{x\to\infty} f(x) = +\infty$, there exists $x_2 \in \mathbb{R}$ such that and for all $x > x_2$, $f(x) > y_0$. One can clearly assume that $x_1 < x_2$ and f being continuous on the sequentially compact interval $[x_1, x_2]$, by the EVT f attains its minimum, say $t \in \mathbb{R}$, at $x_t \in [x_1, x_2]$. Let $m = \min\{y_0, t\} = \min\{f(0), t\}$, then for all $x \in \mathbb{R}$, $f(x) \geq m$ and there exists

$x_m \in \mathbb{R}$ such that $f(x_m) = m$. Indeed, if $t \leq f(0)$ then $m = t$ and we simply take x_m to be x_t. Otherwise, $t > f(0)$ and $m = f(0)$ and we take x_m to be 0. It remains to check that m is actually the infimum of f. Note that m is a lower bound. Assume that r is another lower bound, i.e. for all $x \in \mathbb{R}$, $f(x) \geq r$, then $m = f(x_m) \geq r$ and $m \geq r$. By definition of the infimum, $m = \inf\{f(x) : x \in \mathbb{R}\}$. □

6 Pathological functions

We start with a preliminary result that will be needed in a later exercise.

Exercise 25 (Density of the irrationals in the reals).

1. Prove that $\sqrt{2}$ is irrational.
2. Prove that the irrational are dense in $\mathbb{R}$, i.e. for every real number $x < y$ there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ so that $x < \alpha < y$.

Hint. For (2) use (1). □

Solution.

1. By contradiction.
2. Let $x < y$. Then $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$ and by the density of the rational in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $\frac{x}{\sqrt{2}} < q < \frac{y}{\sqrt{2}}$ and $x < \sqrt{2}q < y$. Let $\alpha = \sqrt{2}q$ then $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (why?) and the conclusion follows.

□

Exercise 26 (A function discontinuous everywhere). The Dirichlet function is defined on $\mathbb{R}$ by

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that every point $x \in \mathbb{R}$ is a point of discontinuity of $\chi_{\mathbb{Q}}$.

Hint. Use the density of the irrationals to prove that every rational is a point of discontinuity. Use the density of the rationals to prove that every irrational is a point of discontinuity. □

Solution. Let $x \in \mathbb{Q}$. By density of the irrationals in $\mathbb{R}$, for every $\delta > 0$, there exists $y \in \mathbb{R} \setminus \mathbb{Q}$ such that $x - \delta < y < x + \delta$ but $|\chi_{\mathbb{Q}}(y) - \chi_{\mathbb{Q}}(x)| = |0 - 1| = 1$. Therefore $\chi_{\mathbb{Q}}$ is not continuous at any rational point.

Let $x \in \mathbb{R} \setminus \mathbb{Q}$. By density of the rationals in $\mathbb{R}$, for every $\delta > 0$, there exists $y \in \mathbb{Q}$ such that $x - \delta < y < x + \delta$ but $|\chi_{\mathbb{Q}}(y) - \chi_{\mathbb{Q}}(x)| = |1 - 0| = 1$. Therefore $\chi_{\mathbb{Q}}$ is not continuous at any irrational point.

□

Exercise 27 (A function discontinuous at every rational point but continuous at every irrational in $(0,1)$). Let g be the Thomae's function, defined on $\mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \text{if } x = 0, \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \setminus \{0\} \text{ (in reduced form).} \end{cases}$$

Prove that g is discontinuous at every rational point but continuous at every irrational in $(0,1)$.

Hint. Use the density of the irrationals to prove discontinuity at every rational. To prove continuity at every irrational in $(0,1)$ consider the set $F = \{\frac{p}{q} \colon 0 < \frac{p}{q} < 1 \text{ and } 2 \leqslant q \leqslant N\}$ for some well chosen N. □

Solution. Assume $r = \frac{p}{q} \in \mathbb{Q}$ (in reduced form with $q \geq 1$). Then for every $\delta > 0$, there exists $y \in \mathbb{R} \setminus \mathbb{Q}$ such that $r - \delta < y < r + \delta$ but $|g(y) - g(r)| = |0 - \frac{1}{q}| = \frac{1}{q} > 0$. Therefore g is not continuous at any rational point.

Now if $x_0 \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1)$. We will show that $\lim_{x \to x_0} g(x) = 0$ and hence g will be continuous at any irrational point in $(0,1)$ since by definition $g(x_0) = 0$. Let $\varepsilon > 0$, then by the Archimedean Principle there exists $N \in \mathbb{N}$ such that $\frac{1}{N} \leq \varepsilon$. Define $F = \{\frac{p}{q} \colon 0 < \frac{p}{q} < 1 \text{ and } 2 \leqslant q \leqslant N\}$. Remark that F is finite, indeed $F = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$. Therefore $\delta = \min\{|x_0 - z| : z \in F\}$ exists and it is strictly positive. We will show that it is the desired number.

Let $x \in (0,1)$ with $|x - x_0| < \delta$, with $x \neq x_0$. We will show that $|f(x)| < \varepsilon$. If x is irrational, then by definition $f(x) = 0$ and hence the conclusion holds. Otherwise, x is rational, and then $x = \frac{p}{q}$. We will show that $q > N$. If this were not the case, then $x \in \{\frac{p}{q} : 0 < \frac{p}{q} < 1 \text{ and } 2 \leqslant q \leqslant N\} = F$ and therefore $|x - x_0| \geqslant \min\{|x_0 - z| : z \in F\} = \delta$, which is absurd. We finally conclude that $|f(x)| = |1/q| \leqslant 1/N < \varepsilon$.

□

Exercise 28 (A function discontinuous at every rational point but continuous at every irrational point). Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ be an enumeration of the set of rational numbers (i.e. for each $q \in \mathbb{Q}$ there is exactly one $n \in \mathbb{N}$ such that $q = q_n$). Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the rule

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{n} & \text{if } x \in \mathbb{Q} \text{ and } x = q_n. \end{cases}$$

Prove that f is continuous at x if and only if $x \in \mathbb{R} \setminus \mathbb{Q}$.

Hint. Show that for all $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ we have $\lim_{x \to x_0} f(x) = 0$. You might want to consider the set $F_N = \{q_n : 1 \leqslant n \leqslant N\}$ for some well chosen N. □

Solution. We will show that for all $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ we have $\lim_{x \to x_0} f(x) = 0$. By the definition of f, this will yield that $f(x_0) = \lim_{x \to x_0} f(x)$ and thus f will be continuous at every irrational point.

Fix $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ and let $\varepsilon > 0$. We will find $\delta > 0$ so that for all $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta$ we have $|f(x) - 0| < \varepsilon$. By the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ with $N \in \mathbb{N}$ and $0 < 1/N < \varepsilon$.

Define $F_N = \{q_n : 1 \leqslant n \leqslant N\}$. Then F_N is a non-empty finite set not containing x_0 and therefore $\delta = \min\{|x_0 - z| : z \in F_N\}$ exists and it is strictly positive. We will show that it is the desired number.

Let $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta$, in particular $x \neq x_0$. We will show that $|f(x)| < \varepsilon$. If x is irrational, then by definition $f(x) = 0$ and hence the conclusion holds. Otherwise, x is rational and hence there exists a unique index m with $x = q_m$. We will show that $m > N$. If this were not the case, then $x \in \{q_n : 1 \leqslant n \leqslant N\} \cap (\mathbb{R} \setminus \{x_0\}) = F$ and therefore $|x - x_0| \geqslant \min\{|x_0 - z| : z \in F\} = \delta$, which is absurd. We finally conclude that $|f(x)| = |1/m| \leqslant 1/N < \varepsilon$.

It remains to prove discontinuity at every rational point. Let $x \in \mathbb{Q}$ then $x = q_n$ for some $n \in \mathbb{N}$. Then for every $\delta > 0$, there exists $y \in \mathbb{R} \setminus \mathbb{Q}$ such that $x - \delta < y < x + \delta$ (by density of the irrationals) but $|g(y) - g(x)| = |0 - \frac{1}{n}| = \frac{1}{n} > 0$. Therefore g is not continuous at any rational point. $\square$