MATH 409, Summer 2019,
Practice Problem Set 4

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1 Differentiability

Exercise 1. Let $x_0 \in (a,b)$ and $f,g\colon (a,b) \to \mathbb{R}$ differentiable at x_0. Show that $f+g$ is differentiable at x_0 and $(f+g)^{'}(x_0) = f^{'}(x_0) + g^{'}(x_0)$.

Hint. Exploit the definition. □

Solution. Assume that f and g are differentiable at x_0. If $h \neq 0$, then

$$\begin{aligned}\frac{(f+g)(x_0+h)-(f+g)(x_0)}{h} &= \frac{f(x_0+h)+g(x_0+h)-(f(x_0)+g(x_0))}{h}\\ &= \frac{f(x_0+h)-f(x_0)+g(x_0+h)-g(x_0))}{h}\\ &= \frac{f(x_0+h)-f(x_0)}{h} + \frac{g(x_0+h)-g(x_0))}{h}.\end{aligned}$$

But $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = f^{'}(x_0)$ and $\lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h} = g^{'}(x_0)$ by assumption. Therefore, $\frac{(f+g)(x_0+h)-(f+g)(x_0)}{h}$ has a limit when h tends to 0 and $(f+g)$ is differentiable at x_0. Moreover,

$$\begin{aligned}(f+g)^{'}(x_0) &= \lim_{h\to 0}\frac{(f+g)(x_0+h)-(f+g)(x_0)}{h}\\ &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} + \lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h}\\ &= f^{'}(x_0)+g^{'}(x_0).\end{aligned}$$

□

Exercise 2. Let $\lambda \in \mathbb{R}$, $x_0 \in (a,b)$ and $f,g\colon (a,b) \to \mathbb{R}$ differentiable at x_0. Show that λf is differentiable at x_0 and $(\lambda f)^{'}(x_0) = \lambda f^{'}(x_0)$.

Hint. Exploit the definition. □

Solution. Assume that f is differentiable at x_0. If $h \neq 0$, then

$$\begin{aligned}\frac{(\lambda \cdot f)(x_0+h) - (\lambda \cdot f)(x_0)}{h} &= \frac{\lambda f(x_0+h) - \lambda f(x_0)}{h} \\ &= \lambda \frac{f(x_0+h) - f(x_0)}{h}.\end{aligned}$$

But $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = f^{'}(x_0)$ by assumption. Therefore, $\frac{(\lambda \cdot f)(x_0+h)-(\lambda \cdot f)(x_0)}{h}$ has a limit when h tends to 0 and $(\lambda \cdot f)$ is differentiable at x_0. Moreover,

$$\begin{aligned}(\lambda \cdot f)^{'}(x_0) &= \lim_{h \to 0} \frac{(\lambda \cdot f)(x_0+h) - (\lambda \cdot f)(x_0)}{h} \\ &= \lim_{h \to 0} \lambda \frac{f(x_0+h) - f(x_0)}{h} \\ &= \lambda f^{'}(x_0).\end{aligned}$$

□

Exercise 3. Let $x_0 \in (a, b)$ and $f, g \colon (a, b) \to \mathbb{R}$ differentiable at x_0. Show that $f \cdot g$ is differentiable at x_0 and $(f \cdot g)^{'}(x_0) = f^{'}(x_0)g(x_0) + f(x_0)g^{'}(x_0)$.

Hint. Exploit the definition. □

Solution. Assume that f and g are differentiable at x_0. If $h \neq 0$, then

$$\begin{aligned}\frac{(f \cdot g)(x_0+h) - (f \cdot g)(x_0)}{h} &= \frac{f(x_0+h)g(x_0+h) - (f(x_0)g(x_0))}{h} \\ &= \frac{(f(x_0+h) - f(x_0))g(x_0+h) + f(x_0)(g(x_0+h) - g(x_0))}{h} \\ &= \frac{f(x_0+h) - f(x_0)}{h} g(x_0+h) + f(x_0) \frac{g(x_0+h) - g(x_0)}{h}.\end{aligned}$$

But $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = f^{'}(x_0)$, $\lim_{h \to 0} \frac{g(x_0+h)-g(x_0)}{h} = g^{'}(x_0)$ and $\lim_{h \to 0} g(x_0+h) = g(x_0)$ by assumption. Therefore, $\frac{(f \cdot g)(x_0+h)-(f \cdot g)(x_0)}{h}$ has a limit when h tends to 0 and $(f \cdot g)$ is differentiable at x_0. Moreover,

$$\begin{aligned}(f \cdot g)^{'}(x_0) &= \lim_{h \to 0} \frac{(f \cdot g)(x_0+h) - (f \cdot g)(x_0)}{h} \\ &= \lim_{h \to 0} [\frac{f(x_0+h) - f(x_0)}{h} g(x_0+h)] + f(x_0) \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h} \\ &= f^{'}(x_0)g(x_0) + f(x_0)g^{'}(x_0).\end{aligned}$$

□

Exercise 4. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and injective on $[a,b]$. Let $g\colon f([a,b]) \to [a,b]$ be the inverse of f onto its image. Show that if $x_0 \in (a,b)$ and f is differentiable at x_0 with $f^{'}(x_0) \neq 0$, then g is differentiable at $f(x_0)$ and $g^{'}(f(x_0)) = \frac{1}{f'(x_0)}$.

Hint: Use the sequential characterization limit and the fact the g is continuous. □

Exercise 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $T \in \mathbb{R}$ (i.e. $f(x) = f(x+T)$ for all $x \in \mathbb{R}$). If f is differentiable, show that f' is periodic with period T.

Hint: Exploit the definition of differentiability together with the periodicity property. □

2 Differentiability and uniform continuity

There are functions that are differentiable on some interval but not uniformly continuous. Under certain circumstances uniform continuity can be proven from differentiability and various addictional assumptions. Here are two examples.

Exercise 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f' : \mathbb{R} \to \mathbb{R}$ is bounded, show that f is uniformly continuous.

Hint: You could show that f is Lipschitz. □

Exercise 7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions so that $g'(x) \neq 0$ for all $x \in \mathbb{R}$ and the function $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = f'(x)/g'(x)$ is bounded. If g is uniformly continuous, show that f is uniformly continuous as well.

Hint: You could observe (and justify) that $|f(x)-f(y)| = \frac{|f(x)-f(y)|}{|g(x)-g(y)|}|g(x)-g(y)|$ and use Cauchy's mean value theorem. □

3 Applications of the Mean Value Theorem

Exercise 8. Let $f\colon (a,b) \to \mathbb{R}$. If f is differentiable on (a,b) with $f^{'}(x) = 0$ for all $x \in (a,b)$, show that there exists $\alpha \in \mathbb{R}$ such that $f(x) = \alpha$ for all $x \in (a,b)$.

Hint. Use the Mean Value Theorem. □

Solution. Assume that $f\colon (a,b) \to \mathbb{R}$ is differentiable on (a,b) with $f^{'}(x) = 0$ for all $x \in (a,b)$. Let $x_1 < x_2$ in (a,b). Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), by the MVT, there exists $x_0 \in (x_1, x_2)$ such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f^{'}(x_0) = 0$ and thus $f(x_1) = f(x_2)$. Pick $c \in (a,b)$ and let $x \in (a,b)$, $x \neq c$ then either $x > c$ or $x < c$. In either case the above argument shows that $f(x) = f(c)$. If we set $\alpha = f(c)$ then it follows that for all $x \in (a,b)$, $f(x) = \alpha$. □

Exercise 9 (Decreasing function test). Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f^{'}(x) \leq 0$ for all $x \in (a,b)$, then f is decreasing on $[a,b]$. If $f^{'}(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing on $[a,b]$.

Hint: Mimic the proof of the increasing function test. □

Possible solution. Let $x_1 < x_2$ in $[a,b]$ then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the MVT there exists $c \in (x_1, x_2)$ such that $f^{'}(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. But $f^{'}(c) \leq 0$ and thus $f(x_2) - f(x_1) \leq 0$. Therefore for every $x_1 < x_2$ in $[a,b]$ one has $f(x_1) \geq f(x_2)$ and f is decreasing. In the case where $f^{'}(c) < 0$, then for every $x_1 < x_2$ in $[a,b]$ one has $f(x_1) > f(x_2)$ and f is strictly decreasing. □

Exercise 10 (First Derivative Test (existence of local maxima)). Let $f\colon (a,b) \to \mathbb{R}$ be differentiable on (a,b). Suppose $c \in (a,b)$ has the property that there exists $\delta > 0$ such that

1. $f^{'}(x)$ exists and $f^{'}(x) \geq 0$ for all $x \in (c-\delta, c) \subseteq (a,b)$, and
2. $f^{'}(x)$ exists and $f^{'}(x) \leq 0$ for all $x \in (c, c+\delta) \subseteq (a,b)$.

Show that f has a local maximum at c.

Hint: Mimic the proof of the First Derivative Test (existence of local minima). □

Exercise 11 (Taylor's Theorem for $n = 2$.). Let $x_0 \in (a,b)$ and $f\colon (a,b) \to \mathbb{R}$. If f is 3 times differentiable on (a,b) and if $x \in (a,b) \setminus \{x_0\}$, then there exists $c_x \in (a,b) \setminus \{x_0\}$ such that

$$f(x) = f(x_0) + f^{'}(x_0)(x-x_0) + \frac{f^{''}(x_0)}{2}(x-x_0)^2 + \frac{f^{(3)}(c_x)}{6}(x-x_0)^3.$$

Hint. Consider the functions $g(t) = f(x) - f(t) - f^{'}(t)(x-t) - \frac{f^{''}(t)}{2}(x-t)^2$ and $h(t) = g(t) - (\frac{x-t}{x-x_0})^3 g(x_0)$ and apply Rolle's Theorem. □

Solution. Let $x > x_0$. Note that $g(x) = 0$ and thus $h(x) = 0$. Moreover, $h(x_0) = g(x_0) - g(x_0) = 0$ and $h^{'}(t) = g^{'}(t) + 3\frac{1}{x-x_0}(\frac{x-t}{x-x_0})^2 g(x_0) = -f^{'}(t) - f^{''}(t)(x-t) + f^{'}(t) - \frac{f^{(3)}(t)}{2}(x-t)^2 + 2\frac{f^{''}(t)}{2}(x-t) + \frac{3}{x-x_0}(\frac{x-t}{x-x_0})^2 g(x_0) = \frac{3}{x-x_0}(\frac{x-t}{x-x_0})^2 g(x_0) - \frac{f^{(3)}(t)}{2}(x-t)^2$. By Rolle's Theorem there exists $c_x \in (x_0, x)$ such that $h^{'}(c_x) = \frac{3}{x-x_0}(\frac{x-c_x}{x-x_0})^2 g(x_0) - \frac{f^{(3)}(c_x)}{2}(x-c_x)^2 = 0$. Therefore, $f(x) - f(x_0) - f^{'}(x_0)(x - x_0) - \frac{f^{''}(x_0)}{2}(x-x_0)^2 - \frac{f^{(3)}(c_x)}{6}(x-x_0)^3 = 0$ and

$$f(x) = f(x_0) + f^{'}(x_0)(x-x_0) + \frac{f^{''}(x_0)}{2}(x-x_0)^2 + \frac{f^{(3)}(c_x)}{6}(x-x_0)^3.$$

□

Exercise 12. If $0 < \alpha \leq 1$, show that $(1+x)^\alpha \leq 1 + \alpha x$ for all $x \in [-1, \infty)$.

Hint. Consider the function $f(t) = t^\alpha$, $t \in [0, \infty)$. □

Possible solution. Since $f^{'}(t) = \alpha t^{\alpha-1}$, it follows from the MVT that $f(1+x) - f(1) = f^{'}(c) = \alpha x c^{\alpha-1}$ for some $c \in (1, 1+x)$. If $x > 0$, then $c > 1$ and since $\alpha \geq 1$ implies $\alpha - 1 \geq 0$, it follows that $c^{\alpha-1} \geq 1$, hence $xc^{\alpha-1} \geq x$. Therefore, we have that $(1+x)^\alpha = f(1+x) = f(1) + \alpha x c^{\alpha-1} \geq f(1) + \alpha x = 1 + \alpha x$.

If $-1 \leq x \leq 0$, then $c \leq 1$ so $c^{\alpha-1} \geq 1$, hence $xc^{\alpha-1} \leq x$ as before since $x \leq 0$. Therefore, we still have that $(1+x)^\alpha = f(1+x) = f(1) + \alpha x c^{\alpha-1} \leq f(1) + \alpha x = 1 + \alpha x$. □

Exercise 13. If $\alpha \geq 1$, show that $(1+x)^\alpha \geq 1 + \alpha x$ for all $x \in [-1, \infty)$.

Hint. Consider the function $f(t) = t^\alpha$, $t \in [0, \infty)$. □

Possible solution. Let $f(t) = t^\alpha$, $t \in [0, \infty)$. Since $f^{'}(t) = \alpha t^{\alpha-1}$, it follows from the MVT that $f(1+x) - f(1) = f^{'}(c) = \alpha x c^{\alpha-1}$ for some $c \in (1, 1+x)$. If $x > 0$, then $c > 1$ and since $0 < \alpha \leq 1$ implies $\alpha - 1 \leq 0$, it follows that $c^{\alpha-1} \leq 1$, hence $xc^{\alpha-1} \leq x$. Therefore, we have that $(1+x)^\alpha = f(1+x) = f(1) + \alpha x c^{\alpha-1} \leq f(1) + \alpha x = 1 + \alpha x$.

If $-1 \leq x \leq 0$, then $c \leq 1$ so $c^{\alpha-1} \leq 1$, hence $xc^{\alpha-1} \geq x$ as before since $x \leq 0$. Therefore, we still have that $(1+x)^\alpha = f(1+x) = f(1) + \alpha x c^{\alpha-1} \geq f(1) + \alpha x = 1 + \alpha x$. □

4 Other applications

Exercise 14 (Bernoulli's Inequality and approximation of e). Let α be a positive real number.

1. Let $(x_n)_{n=1}^{\infty}$ be the sequence where $x_n = (1+\frac{1}{n})^n$ for all $n \in \mathbb{N}$. Show that $(x_n)_{n=1}^{\infty}$ is increasing.
2. Show that $(x_n)_{n=1}^{\infty}$ is bounded above by 3.
3. Show that $(x_n)_{n=1}^{\infty}$ is convergent to $\ell \in [2,3]$.
4. Show that $\lim_{x\to 0^+} \frac{\ln(1+x)}{x} = 1$.
5. Show that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

Hint.

1. You could use Exercise 12 with $x = \frac{1}{n}$ and $\alpha = \frac{n}{n+1}$.
2. You could use the Binomial Formula $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ and the fact that $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$.
3. You could use (1) and (2).
4. You could use L'Hôpital's Rule.
5. You could use the sequential characterization of limits and (4).

□

Solution.

1. Let $n \in \mathbb{N}$, and note that by (1) with $x = \frac{1}{n}$ and $\alpha = \frac{n}{n+1}$, $(1+\frac{1}{n})^{\frac{n}{n+1}} \leq (1+\frac{n}{n+1}\frac{1}{n}) = (1+\frac{1}{n+1})$. Therefore $(1+\frac{1}{n})^n \leq (1+\frac{1}{n+1})^{n+1}$, i.e. $(x_n)_{n=1}^{\infty}$ is increasing.
2. By the Binomial Formula $(1+\frac{1}{n})^n = \sum_{k=0}^{n}\binom{n}{k}(\frac{1}{n})^k 1^{n-k}$, but $\binom{n}{k}(\frac{1}{n})^k = \frac{n(n-1)\dots(n-k+1)}{n^k}\frac{1}{k!} \leq \frac{1}{k!} \leq \frac{1}{2^{k-1}}$ for all $k \in \mathbb{N}$. And thus, $2 = (1+\frac{1}{1}) < (1+\frac{1}{n})^n \leq 1 + 1 + \sum_{k=1}^{n-1}\frac{1}{2^k} = 3 - \frac{1}{2^{n-1}} < 3$.
3. By (1) and (2), $(x_n)_{n=1}^{\infty}$ is increasing and bounded above by 3. Since by (3) $x_n \geq 2$, it follows from the MCT that $(x_n)_{n=1}^{\infty}$ is convergent to $\ell \in [2,3]$.
4. Note that $\lim_{x\to 0^+}\ln(1+x) = \lim_{x\to 0^+} x = 0$, and $x \mapsto f(x) = \ln(1+x)$ and $x \mapsto g(x) = x$ are differentiable on $(0,\infty)$ with $g^{'}(x) = 1 \neq 0$. Since $f^{'}(x) = \frac{1}{1+x}$ and $\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1} = 1$. It follows from L'Hôpital's Rule, that $\lim_{x\to 0^+}\frac{\ln(1+x)}{x} = 1$.
5. Note that $\ln[(1+\frac{1}{n})^n] = n\ln(1+\frac{1}{n}) = \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}$ and by the sequential characterization of limits and (4) $\lim_{n\to\infty}\ln[(1+\frac{1}{n})^n] = 1$. Since the function $x \mapsto e^x$ is continuous on $\mathbb{R}$ and by the sequential characterization of limits one has that $\lim_{n\to\infty} e^{\ln[(1+\frac{1}{n})^n]} = \lim_{n\to\infty}(1+\frac{1}{n})^n = e^1 = e$.

□

The next exercise is Darboux's Theorem, which can be seen as an intermediate value theorem for derivatives (which might not be continuous!).

Exercise 15 (Darboux's Theorem). Let $f : [a, b] \to \mathbb{R}$ be a differentiable function with $f'(a) < f'(b)$. Show that for all $y \in (f'(a), f'(b))$ there exists $c \in (a, b)$ such that $y = f'(c)$.

Hint. You could consider the function $\varphi(t) = f(t) - ty$, and invoke the EVT, together with Fermat's Theorem. □

Exercise 16 (Extension of the derivative). Let $x_0 \in (a, b)$, and $f : (a, b) \to \mathbb{R}$ be a continuous on (a, b) and differentiable on $(a, b) \setminus \{x_0\}$. Let $\lambda \in \mathbb{R}$. If $\lim_{x \to x_0} f'(x) = \lambda$, prove that f' is differentiable at x_0 and $f'(x_0) = \lambda$.

Hint. You could use the MVT. □