REAL ANALYSIS MATH 607 MIDTERM REVIEW PROBLEMS

Problem 1. *You need to know the definitions of:*

(1) *equivalence relations, partial orderings...*
(2) *limsup, liminf for sets and functions...*
(3) *algebras, σ-algebras, π-systems, elementary systems, Dynkin systems...*
(4) *premeasures, outer measures, measures, σ-finite measures, completness of an algebra...*
(5) *simple functions, measurable functions...*

and the proofs of

(1) *Dynkin theorem and the lemmas leading to it,*
(2) *the fundamental approximation lemma of measurable functions by simple functions,*
(3) *the basic properties of the integral of nonnegative simple/measurable functions,*
(4) *the monotone convergence theorem,*
(5) *Fatou's lemma.*

Problem 2. (1) *Let X be a set and $\mathscr{E} = \{\{x\}\colon x \in X\}$. Show that*

$$\mathcal{M}(\mathscr{E}) = \{A \in \mathscr{P}(X)\colon A \text{ is countable or } A^c \text{ is countable}\}.$$

Problem 3. *A family of sets $\mathcal{R} \subset \mathscr{P}(X)$ is called a* ring *if it is non-empty and closed under finite unions and differences. A ring that is closed under countable unions is called a σ-*ring*. Show the following assertions.*

(1) *Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.*
(2) *If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.*
(3) *If $\mathcal{R}$ is a σ-ring, then $\{E \subset X\colon E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.*
(4) *If $\mathcal{R}$ is a σ-ring, then $\{E \subset X\colon E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.*

Problem 4. *Show that every σ-algebra has either finite or uncountably many elements.*

Problem 5. *Assume that the algebra $\mathcal{A}$ generates the σ-algebra $\mathcal{M}$ and assume that μ is a finite measure on $\mathcal{M}$. Show that for any $\varepsilon > 0$ and any $A \in \mathcal{M}$ there is an $\tilde{A} \in \mathcal{A}$ so that $\mu(A\Delta\tilde{A}) < \varepsilon$.*

Problem 6. *Let X be a nonempty set. A class $\mathscr{C} \subset \mathscr{P}(X)$ is* monotone *if*

(1) $(A_n)_{n\geqslant 1} \subset \mathscr{C}$, *$A_n \subseteq A_{n+1}$ for all $n \geqslant 1 \implies \cup_{n=1}^{\infty} A_n \in \mathscr{C}$.*
(2) $(A_n)_{n\geqslant 1} \subset \mathscr{C}$, *$A_n \supseteq A_{n+1}$ for all $n \geqslant 1 \implies \cap_{n=1}^{\infty} A_n \in \mathscr{C}$.*

(a) *(5 points) Show that for any $\mathscr{C} \subset \mathscr{P}(X)$ there exists a smallest monotone class, denoted mon($\mathscr{C}$), that contains $\mathscr{C}$.*
(b) *(10 points) Show that a monotone algebra (i.e. an algebra that is monotone) is a σ-algebra.*
(c) *(15 points) Show that if $\mathscr{A}$ is an algebra, then mon($\mathscr{A}$) $= \mathcal{M}(\mathscr{A})$.*

Problem 7. *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$.*

(1) *Show that $|\mu(A \cap B) - \mu(A)\mu(B)| \leqslant \sqrt{\mu(A)(1-\mu(A))\mu(B)(1-\mu(B))} \leqslant \frac{1}{4}$.*
(2) *Show that $\mu(A \cap B) - \mu(A)\mu(B) \leqslant \min\{\mu(A)(1-\mu(B)), \mu(B)(1-\mu(A))\}$.*
(3) *Show that $\mu(A)\mu(B) - \mu(A \cap B) \leqslant \min\{\mu(A)(1-\mu(A)), \mu(B)(1-\mu(B))\}$.*

Problem 8. *Assume $(X, \mathcal{M}, \mu)$ is a complete measure space.*

(a) *(10 points) If $f\colon (X, \mathcal{M}, \mu) \to \mathbb{R}$ is measurable and $f = g$ μ-almost everywhere (i.e. $\mu(\{f \neq g\}) = 0$ and we simply write μ-a.e.), then g is also measurable.*
(b) *(10 points) If $f_n\colon (X, \mathcal{M}, \mu) \to \mathbb{R}$ is measurable for $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n = f$ μ-a.e., then f is measurable.*

Problem 9. *Throughout this problem let $f\colon X \to Y$ be a function.*

1. *(5 points) If $\mathcal{M}$ is a σ-algebra on X, is it true that $f(\mathcal{M}) \overset{\text{def}}{=} \{f(A)\colon A \in \mathcal{M}\}$ is a σ-algebra on Y?*
2. *(5 points) (Pushforward or direct-image σ-algebra) If $\mathcal{M}$ is a σ-algebra on X, show that $\Sigma \overset{\text{def}}{=} \{B \subset Y\colon f^{-1}(B) \in \mathcal{M}\}$ is a σ-algebra on Y.*
3. *(5 points) (Transport Lemma) Let $\mathscr{C} \subset \mathscr{P}(Y)$. Show that $f^{-1}(\mathcal{M}(\mathscr{C})) = \mathcal{M}(f^{-1}(\mathscr{C}))$.*
4. *(5 points) Let $\mathcal{M}_X$ be a σ-algebra on X and $\mathscr{C} \subset \mathscr{P}(Y)$. Use the transport lemma to show that f is $(\mathcal{M}_X, \mathcal{M}(\mathscr{C}))$-mesurable if $f^{-1}(\mathscr{C}) \subset \mathcal{M}_X$.*
5. *(5 points) (Trace σ-algebra) Show that if $X \subset Y$, $\mathcal{M}$ is a σ-algebra on Y, and f is the identity function, then $f^{-1}(\mathcal{M}) = \{X \cap A\colon A \in \mathcal{M}\}$.*
6. *(5 points) Show that if $X = Y \times Z$, $\mathcal{M}$ is a σ-algebra on Y, and f is the canonical projection from $Y \times Z$ onto Y (i.e., $f(y,z) = y$), then $f^{-1}(\mathcal{M}) = \{A \times Z\colon A \in \mathcal{M}\}$.*

Problem 10. *(1) For all $i \in I$, let $f_i\colon Y \to (Y_i, \mathcal{M}_i)$ where $\mathcal{M}_i$ is a σ-algebra. Let $\mathcal{M}_Y$ be the smallest σ-algebra making every map f_i, $(\mathcal{M}_Y, \mathcal{M}_i)$-measurable. Let $\mathcal{M}_X$ be a σ-algebra on X. Show that a map $f\colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is $(\mathcal{M}_X, \mathcal{M}_Y)$-measurable if and only if for all $i \in I$, $f_i \circ f\colon (X, \mathcal{M}_X) \to (Y_i, \mathcal{M}_i)$ is $(\mathcal{M}_X, \mathcal{M}_i)$-measurable*

Problem 11. *(1) Show that $\Sigma = \{A \in \mathscr{P}(\mathbb{R})\colon A = -A\}$ is a σ-algebra.*
(2) Characterize the measurable functions from $(\mathbb{R}, \Sigma)$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

Problem 12. *Show that a monotone function $f\colon \mathbb{R} \to \mathbb{R}$ is Borel.*

Problem 13. *Let $f\colon (X, \mathcal{A}, \mu) \to (Y, \mathcal{B})$ be measurable. Show that $\mu_f\colon \mathcal{B} \to [0, \infty]$ defined by $\mu_f(B) = \mu(f^{-1}(B))$ is a measure on $\mathcal{B}$.*

Problem 14. *Let $f\colon (X, \mathcal{M}, \mu) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be measurable.*

1. *Show that if $\mu(X) \neq 0$ then there exists $A \in \mathcal{M}$, $\mu(A) \neq 0$, such that f is bounded on A.*
2. *Show that if $\mu(\{f \neq 0\}) \neq 0$ then there exists $A \in \mathcal{M}$, $\mu(A) \neq 0$, and $c > 0$ such that $|f| \geqslant c$ on A.*

Problem 15. *For all $n \geqslant 1$, let $f_n\colon (X, \mathcal{M}, \mu) \to [0, \infty]$ be measurable.*

1. *Show that $\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$.*
2. *Let $(a_{n,m})_{n \in \mathbb{N}, m \in \mathbb{N}} \subset [0, \infty)$. Show that $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{n,m}$.*