MATH 172-501 *Homework #11*

Section 11.3: Use the integral test to determine whether the series is convergent or divergent.

6. The function $f(x) := \frac{1}{(3x-1)^4}$ is continuous, positive and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{1}{(3x-1)^4}dx = \lim_{t\to\infty}\int_1^t (3x-1)^{-4}dx = \lim_{t\to\infty}\left[\frac{(3x-1)^{-3}}{-9}\right|_1^t = \frac{1}{72}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(3n-1)^4} \text{ is convergent.}$$

7. The function $f(x) := \frac{x}{x^2+1}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{x}{x^2+1}dx = \lim_{t\to\infty}\int_1^t \frac{x}{x^2+1}dx = \lim_{t\to\infty}\left[\frac{1}{2}\ln(x^2+1)\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\ln(t^2+1) - \frac{1}{2}\ln(2)\right] = \infty.$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{n}{n^2+1} \text{ is divergent.}$$

8. The function $f(x) := x^2 e^{-x^3}$ is continuous, positive and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty x^2 e^{-x^3}dx = \lim_{t\to\infty}\int_1^t x^2 e^{-x^3}dx = \lim_{t\to\infty}\left[-\frac{1}{3}e^{-x^3}\right|_1^t = \frac{1}{3e}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{n^2}{e^{n^3}} \text{ is convergent}$$

Section 11.4: Determine whether the series converges or diverges.

3. $\frac{1}{n^3+8} < \frac{1}{n^3} \ \forall n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^3+8}$ converges by Comparison Test to $\sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges as a p-series with $p = 3 > 1$.

10. $\frac{k \sin^2(k)}{1+k^3} \leq \frac{k}{1+k^3} < \frac{k}{k^3} = \frac{1}{k^2} \ \forall k \geq 1$, so

$\sum_{k=1}^{\infty} \frac{k \sin^2(k)}{1+k^3}$ converges by Comparison Test to $\sum_{k=1}^{\infty} \frac{1}{k^2}$

which converges as a p-series with $p = 2 > 1$.

13. $\frac{1+\cos(n)}{e^n} < \frac{2}{e^n} \quad \forall n \geq 1.$

$2 \sum_{n=1}^{\infty} \frac{1}{e^n}$ is a convergent geometric series with

$|r| = |\frac{1}{e}| < 1 \implies \sum_{n=1}^{\infty} \frac{1+\cos(n)}{e^n}$ converges by

Comparison Test.

Section 11.5: Test the series for convergence or divergence.

5. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n} = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$. Now,

$b_n = \frac{1}{3+5n} > 0 \ \forall n \geq 1$, $\{b_n\}$ is decreasing and

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3+5n} = 0$. $\therefore$ the series converges

by Alternating Series Test.

9. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n e^{-n} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now, $b_n = e^{-n} > 0 \ \forall n \geq 1$,

$\{b_n\}$ is decreasing and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{e^n} = 0$.

$\therefore$ The series converges by Alternating Series Test.

10. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now, $b_n = \frac{\sqrt{n}}{2n+3} > 0 \ \forall n \geq 1$,

$\{b_n\}$ is decreasing $\forall n \geq \underline{\underline{2}}$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{2n+3} = 0$.

$\therefore$ The series converges by Alternating Series Test.

Section 11.6: Use the Ratio Test to determine whether the series is convergent or divergent.

7. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cdot 5^n}{5^{n+1}\cdot n} \right| = \frac{1}{5} \cdot \lim_{n\to\infty} \left(\frac{n+1}{n} \right) = \frac{1}{5} < 1.$

$\therefore \sum_{n=1}^{\infty} \frac{n}{5^n}$ converges absolutely by Ratio Test.

8. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}\cdot n^2}{(n+1)^2(-2)^n} \right| = 2 \cdot \lim_{n\to\infty} \left(\frac{n^2}{(n+1)^2} \right) = 2 > 1$

$\therefore \sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ diverges by Ratio Test.

12. $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)e^{-(k+1)}}{ke^{-k}} \right| = \frac{1}{e} \cdot \lim_{k\to\infty} \left(\frac{k+1}{k} \right) = \frac{1}{e} < 1$

$\therefore \sum_{k=1}^{\infty} ke^{-k}$ converges absolutely by Ratio Test.