MATH 172-501 Homework Solutions 09/12

Section 6.1:

14. Sketch the region enclosed by the given curves and find its' area.

$y = x^2$, $y = 4x - x^2$

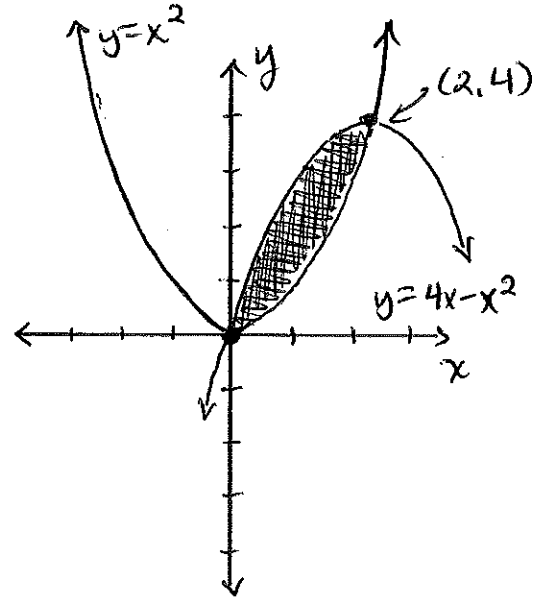
$x^2 = 4x - x^2 \iff 2x^2 - 4x = 0 \iff x = 0 \text{ or } 2.$

$$A = \int_0^2 \left[(4x - x^2) - (x^2)\right] dx$$

$$= \int_0^2 (4x - 2x^2)\,dx$$

$$= \left[2x^2 - \tfrac{2}{3}x^3\right|_0^2$$

$$= \left(8 - \tfrac{16}{3}\right) - (0 - 0) = \tfrac{8}{3}$$



16. Sketch the region enclosed by the given curves and find its' area.

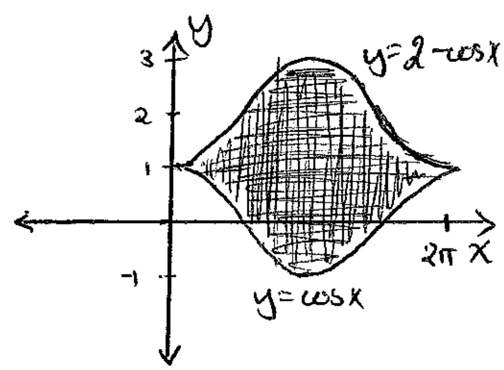
$y = \cos(x)$, $y = 2 - \cos(x)$, $0 \le x \le 2\pi$

$$A = \int_0^{2\pi} \left[(2 - \cos(x)) - (\cos(x))\right] dx$$

$$= \int_0^{2\pi} (2 - 2\cos(x))\,dx$$

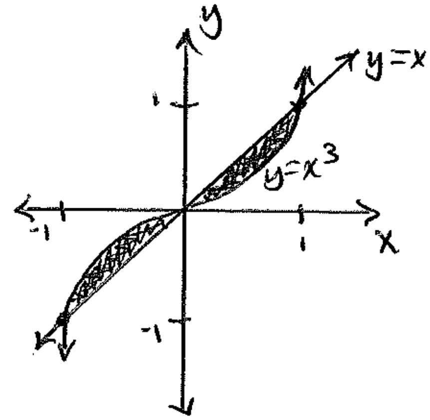
$$= \left[2x - 2\sin(x)\right|_0^{2\pi}$$

$$= (4\pi - 0) - (0 - 0) = 4\pi$$



22. Sketch the region enclosed by the given curves and find its' area.

$y=x^3$, $y=x$

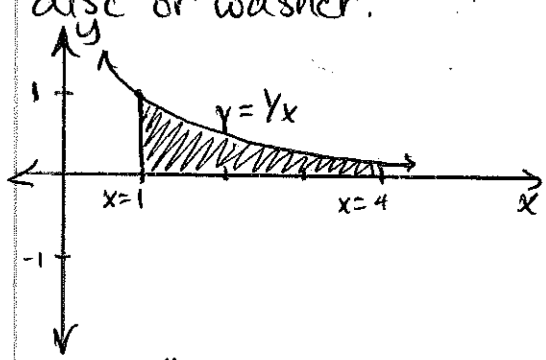


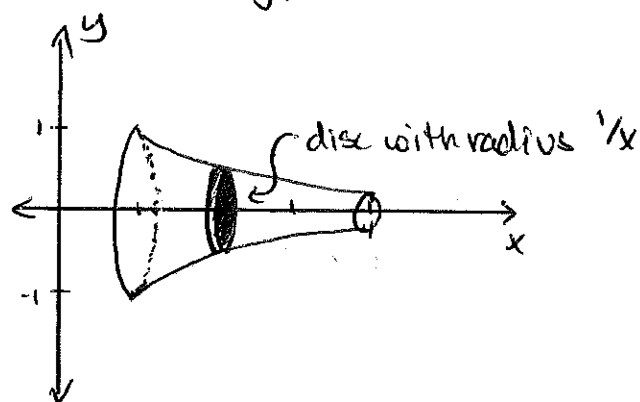
$x^3=x \Leftrightarrow x^3-x=0 \Leftrightarrow$
$x=0$ or ± 1.

$$A = 2\int_0^1 (x-x^3)\,dx$$
$$= 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right|_0^1$$
$$= 2\left[\left(\tfrac{1}{2}-\tfrac{1}{4}\right)-(0-0)\right] = \tfrac{1}{2}$$

Section 6.2:

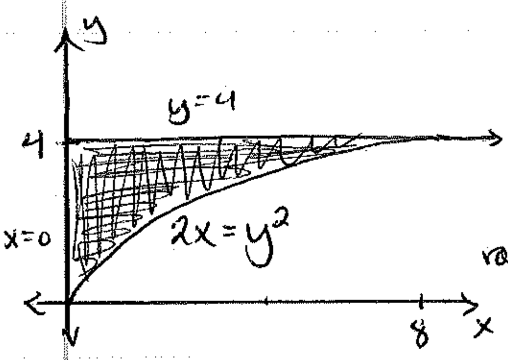
2. Find the volume of the solid generated by rotating the region bounded by the curves $y=1/x$, $y=0$, $x=1$, and $x=4$ about the x-axis. Sketch the region, the solid and a typical disc or washer.

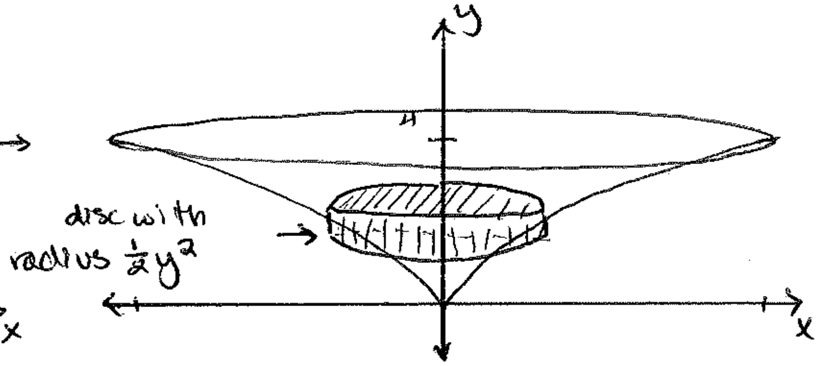




$$V = \int_1^4 \frac{\pi}{x^2}\,dx = \pi\left[-\frac{1}{x}\right|_1^4 = \pi\left[1-\frac{1}{4}\right] = \frac{3\pi}{4}$$

6. Find the volume of the solid generated by rotating the region bounded by the curves $2x=y^2$, $x=0$, and $y=4$ about the y-axis. Sketch the region, the solid, and a typical disc or washer.





$$V=\int_0^4 \left(\frac{\pi}{4}y^4\right)dy = \frac{\pi}{4}\left[\frac{1}{5}y^5\right|_0^4 = \frac{\pi}{20}\left[4^5-0\right]$$

$$= \frac{\pi\cdot 4^4}{5} = \frac{256\pi}{5}$$

14. Find the volume of the solid generated by rotating the region bounded by the curves $y=\sin x$, $y=\cos x$, $0\le x\le \pi/4$ about $y=-1$. Sketch the region, the solid, and a typical disc or washer

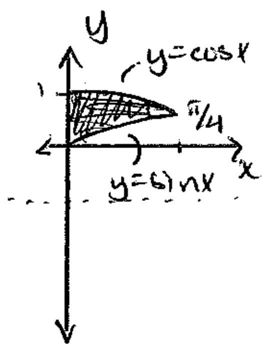
$$V=\int_0^{\pi/4}\pi\left[(\cos(x)+1)^2-(\sin(x)+1)^2\right]dx$$

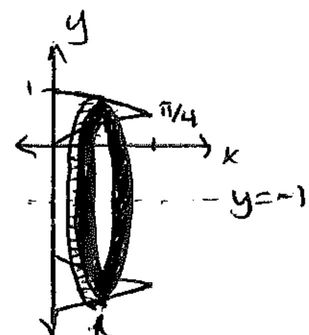
$$=\pi\int_0^{\pi/4}\cos^2x-\sin^2x+2\cos x-2\sin x\,dx$$

$$=\pi\int_0^{\pi/4}\cos(2x)+2\cos x-2\sin x\,dx$$

$$=\pi\left[\frac{1}{2}\sin(2x)+2\sin x+2\cos x\right|_0^{\pi/4}$$

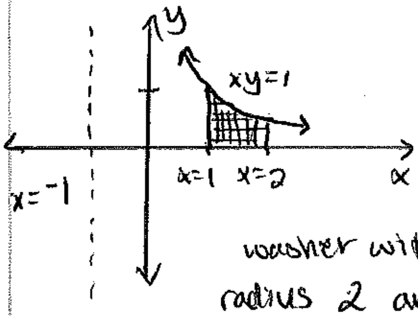
$$=\pi\left[\left(\frac{1}{2}+\sqrt{2}+\sqrt{2}\right)-2\right]=\pi\left[2\sqrt{2}-\frac{3}{2}\right]$$

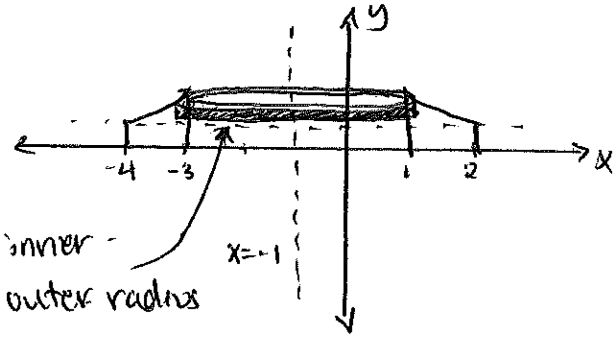




washer with inner radius $(\sin x+1)$ and outer radius $(\cos x+1)$

16. Find the volume of the solid generated by rotating the region bounded by the curves $xy=1$, $y=0$, $x=1$, $x=2$ about $x=-1$.
Sketch the region, the solid, and a typical disc or washer.





washer with inner radius 2 and outer radius $1/y + 1$.

$$V = \pi\int_0^{1/2} (3)^2 - (2)^2 \, dy + \pi\int_{1/2}^{1} (1/y + 1)^2 - (2)^2 \, dy$$

$$= \frac{5\pi}{2} + \pi\int_{1/2}^{1} 1/y^2 + 2/y - 3 \, dy$$

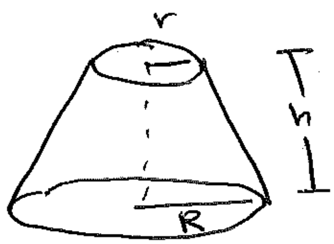
$$= \frac{5\pi}{2} + \pi\left[-\frac{1}{y} + 2\ln|y| - 3y\right|_{1/2}^{1}$$

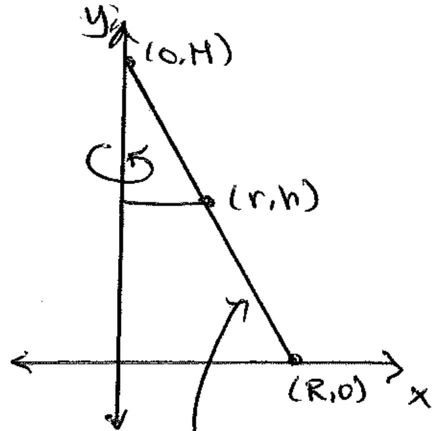
$$= \frac{5\pi}{2} + \pi\left[(-1 + 0 - 3) - (-2 + 2\ln|1/2| - \tfrac{3}{2})\right]$$

$$= \frac{5\pi}{2} + \pi\left(-\frac{1}{2} - 2\ln(1/2)\right) = 2\pi + 2\pi\ln(2)$$

$$= 2\pi(1 + \ln(2))$$

48. Find the volume of a frustum of a right circular cone with height h, lower base radius R and upper radius r.





$$y-h=\frac{h}{R-r}(r-x)$$

$$\Rightarrow x = R - y\left(\frac{R-r}{h}\right)$$

$$V = \pi\int_0^h \left(R - y\left(\frac{R-r}{h}\right)\right)^2 dy = \pi\int_0^h R^2 - 2Ry\left(\frac{R-r}{h}\right) + y^2\left(\frac{R-r}{h}\right)^2 dy$$

$$= \pi\left[R^2 y - R\left(\frac{R-r}{h}\right)y^2 + \left(\frac{R-r}{h}\right)^2 \frac{y^3}{3}\right]_0^h$$

$$= \pi\left[R^2 h - Rh^2\left(\frac{R-r}{h}\right) + \left(\frac{R-r}{h}\right)^2 \frac{h^3}{3}\right]$$

$$= \pi\left[\cancel{R^2 h} - \cancel{R^2 h} + Rrh + \frac{1}{3}h(R-r)^2\right]$$

$$= \pi\left[Rrh + \frac{1}{3}R^2 h - \frac{2}{3}Rrh + \frac{1}{3}r^2 h\right]$$

$$= \frac{\pi h}{3}\left[R^2 + Rr + r^2\right]$$