Section 7.3:

22.

$$\star = \int_0^1 \sqrt{1+x^2}\,dx \qquad \text{Let } x=\tan\theta,\ \theta\in(-\pi/2,\pi/2)$$
$$\Rightarrow dx=\sec^2\theta\,d\theta$$

$$\Rightarrow \star = \int_0^{\pi/4}\sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4}$$

$$= \frac{1}{2}(\sqrt{2}\cdot 1) + \frac{1}{2}\ln|\sqrt{2}+1| - \frac{1}{2}(1\cdot 0) - \frac{1}{2}\ln|1+0|$$

$$= \frac{1}{2}\left[\sqrt{2}+\ln(1+\sqrt{2})\right]$$

Section 7.4:

11.

$$\star := \int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \frac{2}{(2x+1)(x+1)}\,dx$$

$$\frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$

$$\Rightarrow \begin{cases} A+2B=0 \\ A+B=2 \end{cases} \Rightarrow \begin{cases} A+2(2-A)=0 \\ \Rightarrow A=4,\ B=-2 \end{cases}$$

$$\Rightarrow \star = \int_0^1 \frac{4}{2x+1} - \frac{2}{x+1}\,dx = \left[2\ln|2x+1| - 2\ln|x+1|\right]_0^1$$

$$= 2\ln(3) - 2\ln(2) + 2\ln(1) - 2\ln(1) = 2\ln\left(\frac{3}{2}\right)$$

23.

$$\star := \int \frac{10}{(x-1)(x^2+9)}\,dx$$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$\Rightarrow \begin{cases} A+B=0 \\ C-B=0 \\ 9A-C=10 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=-1 \end{cases}$$

$$\Rightarrow \star = \int \frac{1}{x-1} - \frac{x+1}{x^2+9}\,dx$$

$$= \ln|x-1| - \frac{1}{2}\int \frac{2x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{9}\int \frac{1}{(\frac{x}{3})^2+1}\,dx$$

Let $\frac{x}{3} = \tan\theta$, $\theta \in (-\pi/2, \pi/2)$

$\Rightarrow dx = 3\sec^2\theta\,d\theta$

$$\Rightarrow \star = \ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\theta + C$$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

25.

$$\star := \int \frac{4x}{x^3+x^2+x+1}\,dx = \int \frac{4x}{(x+1)(x^2+1)}\,dx$$

$$\frac{4x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \begin{cases} A+B=0 \\ B+C=4 \\ A+C=0 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=2 \\ C=2 \end{cases}$$

$$\Rightarrow \star = \int \frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1}\,dx$$

$$= -2\ln|x+1| + \ln|x^2+1| + 2\int \frac{1}{x^2+1}\,dx$$

Let $x = \tan\theta$, $\theta \in (-\pi/2, \pi/2)$

$\Rightarrow dx = \sec^2\theta\,d\theta$

$$\Rightarrow \star = -2\ln|x+1| + \ln|x^2+1| + 2\int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$$

$$= -2\ln|x+1| + \ln(x^2+1) + 2\theta + C$$

$$= -2\ln|x+1| + \ln(x^2+1) + 2\tan^{-1}(x) + C$$