MATH 171-501 Homework # 8

Section 7.8:

Use the Comparison Theorem to determine whether the integral is convergent or divergent.

49. $\forall x > 0$, $\frac{x}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2}$. $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by the p-test with $p = 2 > 1$, so

$\int_1^\infty \frac{x}{x^3+1}\,dx$ is convergent by comparison theorem.

$\int_0^1 \frac{x}{x^3+1}\,dx$ is some finite constant, so

$\int_0^\infty \frac{x}{x^3+1}\,dx$ is also convergent.

50. $\forall x \geq 1$, $\frac{1+\sin^2 x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent by the p-test with $0 < p = \frac{1}{2} \leq 1$.

$\therefore \int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$ is divergent by comparison theorem.

51. For $x > 1$, $f(x) := \frac{x+1}{\sqrt{x^4 - x}} > \frac{x+1}{\sqrt{x^4}} > \frac{x}{x^2} = \frac{1}{x}$, so

$\int_2^\infty f(x)\,dx$ diverges by comparison theorem with $\int_2^\infty \frac{1}{x}\,dx$

which diverges by p-test with $p = 1 \leq 1$

$\therefore \int_1^\infty f(x)\,dx$ also diverges.

58. Find the values of p for which the integral converges and evaluate the integral for those values of p.

Consider $\int_e^\infty \frac{dx}{x(\ln x)^p}$. Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$\Rightarrow \int_e^\infty \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty} \int_1^{\ln(t)} u^{-p}\,du$$

which converges to $\frac{1}{p-1}$ if $p>1$ and diverges otherwise.

69. $\gamma = \int_0^\infty \frac{cN(1-e^{-kt})}{k} e^{-\lambda t}\,dt$, $c,k,\lambda \in \mathbb{R}^+$, $N \in \mathbb{N}$.

$$\gamma = \frac{cN}{k} \lim_{x\to\infty} \int_0^x \left[e^{-\lambda t} - e^{(-k-\lambda)t}\right] dt$$

$$= \frac{cN}{k} \lim_{x\to\infty} \left[-\frac{1}{\lambda} e^{-\lambda t} + \frac{1}{k+\lambda} e^{-(k+\lambda)t}\right|_0^x$$

$$= \frac{cN}{k} \lim_{x\to\infty} \left[-\frac{1}{\lambda} e^{-\lambda x} + \frac{1}{k+\lambda} e^{-(k+\lambda)x} + \frac{1}{\lambda} - \frac{1}{k+\lambda}\right]$$

$$= \frac{cN}{k}\left[\frac{1}{\lambda} - \frac{1}{k+\lambda}\right] = \frac{cN}{\lambda(k+\lambda)}$$

73. Find the Laplace transforms of the following functions:

(a.)
$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \lim_{n\to\infty}\left[\frac{e^{-ns}}{-s} + \frac{1}{s}\right] \to \frac{1}{s}$$

only if $s>0$. $\therefore F(s) = \frac{1}{s}$ with domain $\{s \mid s>0\}$

(b.)

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^t e^{-st}\,dt = \lim_{n\to\infty}\int_0^n e^{t(1-s)}\,dt$$

$$= \lim_{n\to\infty}\left[\frac{1}{1-s}e^{t(1-s)}\right|_0^n = \lim_{n\to\infty}\left[\frac{1}{1-s}e^{n(1-s)} + \frac{1}{s-1}\right]$$

$\longrightarrow \frac{1}{s-1}$ only if $1-s<0$ i.e. if $s>1$.

$\therefore$ $F(s) = \frac{1}{s-1}$ with domain $\{s \mid s>1\}$.

(c)

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty te^{-st}\,dt = \lim_{n\to\infty}\int_0^n te^{-st}\,dt$$

Let $u=t$ $\quad dv = e^{-st}\,dt$

$du = dt$ $\quad v = -\frac{1}{s}e^{-st}\,dt$

$$\Rightarrow F(s) = \lim_{n\to\infty}\left[\left[-\frac{1}{s}te^{-st}\right|_0^n + \frac{1}{s}\int_0^n e^{-st}\,dt\right]$$

$$= \lim_{n\to\infty}\left[\left[-\frac{1}{s}ne^{-sn} + 0\right] + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right|_0^n\right]$$

$$= \lim_{n\to\infty}\left[\frac{-n}{se^{sn}} - \frac{1}{s^2e^{sn}} + \frac{1}{s^2}\right] \Longrightarrow \frac{1}{s^2} \text{ iff } s>0.$$

$\therefore$ $F(s) = \frac{1}{s^2}$ with domain $\{s \mid s>0\}$.

74. $0 \le f(t) \le Me^{at} \Rightarrow 0 \le f(t)e^{-st} \le Me^{(a-s)t}$ $\forall t \ge 0$.

Using comparison theorem:

$$\int_0^\infty Me^{(a-s)t}\,dt = \lim_{n\to\infty} M\int_0^n e^{(a-s)t}\,dt = \lim_{n\to\infty}\frac{1}{a-s}\left[e^{(a-s)t}\right|_0^n$$

$= \frac{M}{a-s}\cdot\lim_{n\to\infty}\left[e^{(a-s)n} - 1\right]$ which converges $\forall s>a$.

$\therefore$ $F(s)$ converges $\forall s>a$.

80. $I = \int_0^\infty \left(\frac{x}{x^2+1} - \frac{c}{3x+1} \right) dx = \lim_{t\to\infty} \left[\frac{1}{2}\ln(x^2+1) - \frac{c}{3}\ln(3x+1) \right]_0^t$

$= \lim_{t\to\infty} \left[\frac{1}{2}\ln(t^2+1) - \frac{c}{3}\ln(3t+1) - 0 + 0 \right]$

$= \lim_{t\to\infty} \ln\left[(t^2+1)^{1/2} \right] - \ln\left[(3t+1)^{c/3} \right]$

$= \lim_{t\to\infty} \ln \left[\frac{\sqrt{t^2+1}}{(3t+1)^{c/3}} \right] = \ln \left[\lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{c/3}} \right]$

For $c \le 0$, the integral diverges.

For $c > 0$,

$$L = \lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{c/3}} \overset{L'H}{=} \lim_{t\to\infty} \frac{t/\sqrt{t^2+1}}{c(3t+1)^{(c/3)-1}} = \frac{1}{c} \cdot \lim_{t\to\infty} \frac{1}{(3t+1)^{(c/3)-1}}$$

For $c/3 < 1$, $L = \infty$ and I diverges

For $c = 3$, $L = \frac{1}{3}$ and $I = \ln\left(\frac{1}{3}\right)$

For $c > 3$, $L = 0$ and I diverges to $-\infty$.

$\therefore$ The integral converges if $c = 3$.