

MATH 172 Homework 4

6.4: #4 ; 6.5: # 6, 26 ;

7.1: # 6, 18, 40, 48, 52, 64, 70

6.4

$$W = \int_{1}^2 \cos\left(\frac{\pi x}{3}\right) dx \quad u = \frac{\pi x}{3}$$

$$du = \frac{\pi}{3} dx$$

$$dx = \frac{3}{\pi} du$$

$$@ x=1, u=\frac{\pi}{3}$$

$$@ x=2, u=\frac{2\pi}{3}$$

$$= \int_{\pi/3}^{2\pi/3} \cos u \left(\frac{3}{\pi} du\right) = \frac{3}{\pi} \int_{\pi/3}^{2\pi/3} \cos u du$$

$$= \frac{3}{\pi} \sin u \Big|_{\pi/3}^{2\pi/3} = \frac{3}{\pi} \left(\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right) \right)$$

$$= \frac{3}{\pi} \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) = \boxed{0}$$

From $x=1$ to $x=3/2$, the force does work equal to

$$W = \int_{1}^{3/2} \cos\left(\frac{\pi}{3}x\right) dx = \frac{3}{\pi} \sin u \Big|_{\pi/3}^{\pi/2}$$

$= \frac{3}{\pi} \left(1 - \frac{\sqrt{3}}{2} \right) J$ which accelerates the particle and increases the kinetic energy.

From $x=3/2$ to $x=2$, the force acts in the opposite direction of the motion of the particle.

$$W = \int_{3/2}^2 \cos\left(\frac{\pi}{3}x\right) dx = \frac{3}{\pi} \left(\frac{\sqrt{3}}{2} - 1 \right) J.$$

This decreases the kinetic energy of the particle and does negative work that is equal in magnitude and opposite in sign to the work done from $x=1$ to $x=3/2$.

6.5

6. $f(x) = \frac{x^2}{(x^3 + 3)^2}, [-1, 1]$

$$\text{fave} = \frac{1}{1 - (-1)} \int_{-1}^1 \frac{x^2}{(x^3 + 3)^2} dx \quad u = x^3 + 3 \\ du = 3x^2 dx \quad dx = \frac{du}{3x^2}$$

$$@ x = -1, u = 2$$

$$@ x = 1, u = 4$$

$$= \frac{1}{2} \int_2^4 \frac{x^2}{u^2} \left(\frac{du}{3x^2} \right)$$

$$= \frac{1}{6} \int_2^4 u^{-2} du = -\frac{1}{6} \cdot u^{-1} \Big|_2^4$$

$$= -\frac{1}{6} \left(\frac{1}{4} - \frac{1}{2} \right) = -\frac{1}{6} \left(-\frac{1}{4} \right) = \boxed{\frac{1}{24}}$$

26. Show

$$\text{fave}[a, b] = \underbrace{\frac{c-a}{b-a} \text{fave}[a, c]}_{\textcircled{A}} + \frac{b-c}{b-a} \text{fave}[c, b]$$

$$\textcircled{A} = \frac{c-a}{b-a} \cdot \frac{1}{c-a} \int_a^c f(x) dx + \frac{b-c}{b-a} \cdot \frac{1}{b-c} \int_c^b f(x) dx$$

$$= \frac{1}{b-a} \int_a^c f(x) dx + \frac{1}{b-a} \int_c^b f(x) dx$$

$$= \frac{1}{b-a} \left[\int_a^c f(x) dx + \int_c^b f(x) dx \right]$$

$$= \frac{1}{b-a} \int_a^b f(x) dx = \text{fave}[a, b] \quad \boxed{\square}$$

7.1

6. $\int (x-1) \sin \pi x \, dx$

Let $u = x - 1$ $dv = \sin \pi x \, dx$
 $du = dx$ $v = -\frac{1}{\pi} \cos \pi x$

$$= -\frac{1}{\pi} (x-1) \cos \pi x - \int -\frac{1}{\pi} \cos \pi x \, dx$$

$$= -\frac{1}{\pi} (x-1) \cos \pi x + \frac{1}{\pi} \left(\frac{1}{\pi} \sin \pi x \right) + C$$
$$= \boxed{-\frac{1}{\pi} (x-1) \cos \pi x + \frac{1}{\pi^2} \sin \pi x + C}$$

18. $\int e^{-\theta} \cos 2\theta \, d\theta = I$

Let $u = e^{-\theta}$ $dv = \cos 2\theta \, d\theta$
 $du = -e^{-\theta} d\theta$ $v = \frac{1}{2} \sin 2\theta$

(*) $I = \frac{1}{2} e^{-\theta} \sin 2\theta - \int -e^{-\theta} \left(\frac{1}{2} \sin 2\theta \right) d\theta$
 $= \frac{1}{2} e^{-\theta} \sin 2\theta + \frac{1}{2} \int e^{-\theta} \sin 2\theta \, d\theta$

Let $u = e^{-\theta}$ $dv = \sin 2\theta \, d\theta$
 $du = -e^{-\theta} d\theta$ $v = -\frac{1}{2} \cos 2\theta$

$$\int e^{-\theta} \sin 2\theta \, d\theta = -\frac{1}{2} e^{-\theta} \cos 2\theta - \int -e^{-\theta} \left(-\frac{1}{2} \cos 2\theta \right) d\theta$$
$$= -\frac{1}{2} e^{-\theta} \cos 2\theta - \frac{1}{2} \int e^{-\theta} \cos 2\theta \, d\theta$$

"I"

From (*)

$$I = \frac{1}{2} e^{-\theta} \sin 2\theta + \frac{1}{2} \left(-\frac{1}{2} e^{-\theta} \cos 2\theta - \frac{1}{2} I \right)$$

$$I = \frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta - \frac{1}{4} I$$

$$\frac{5}{4} I = \frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta$$

$$I = \frac{4}{5} \left(\frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta \right) + C$$

$$I = \boxed{\frac{2}{5} e^{-\theta} \sin 2\theta - \frac{1}{5} e^{-\theta} \cos 2\theta + C}$$

$$40. \int_0^{\pi} e^{\cos t} \sin 2t dt = \int_0^{\pi} e^{\cos t} (2 \sin t \cos t) dt$$

$$\text{Let } x = \cos t \Rightarrow dx = -\sin t dt \\ \Rightarrow dt = \frac{-dx}{\sin t}$$

$$@ t=0, x=1 ; @ t=\pi, x=-1$$

$$= \int_{-1}^1 e^x (2 \sin t) \left(-\frac{dx}{\sin t} \right)$$

$$= -2 \int_{-1}^1 x e^x dx = 2 \int_{-1}^1 x e^x dx$$

$$\text{Let } u=x \quad dv = e^x dx$$

$$du = dx \quad v = e^x$$

$$= 2 \left[x e^x \Big|_{-1}^1 - \int e^x dx \right]$$

$$= 2 \left[e + e^{-1} - e^x \Big|_{-1}^1 \right] = 2 \left[e + \frac{1}{e} - (e - \frac{1}{e}) \right]$$

$$= 2(e + \frac{1}{e} - e + \frac{1}{e}) = \boxed{\frac{4}{e}}$$

48.

a) $\int \cos^n x dx = \int \cos^{n-1} x \cos x dx$

Let $u = \cos^{n-1} x$ $dv = \cos x dx$
 $du = -(n-1) \cos^{n-2} x \sin x dx$
 $v = \sin x$

$$\int \cos^n x dx = \sin x \cos^{n-1} x$$

$$+ \underbrace{\int (n-1) \cos^{n-2} x \sin^2 x dx}_{\textcircled{K}}$$

$$\textcircled{K} = (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx$$

$$= (n-1) \int (\cos^{n-2} x - \cos^n x) dx$$

$$= (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

$$= (n-1) \int \cos^{n-2} x dx - n \int \cos^n x dx + \int \cos^n x dx$$

Substitute:

$$\int \cos^n x dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x dx$$

$$= n \int \cos^n x dx + \int \cos^n x dx$$

Rearrange:

$$n \int \cos^n x dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

b) Let $n = 2$

$$\begin{aligned}\int \cos^2 x dx &= \frac{1}{2} \cos x \sin x + \frac{1}{2} \int dx \\ &= \frac{1}{2} \cos x \sin x + \frac{1}{2} x + C\end{aligned}$$

$$= \boxed{\frac{\sin 2x}{4} + \frac{1}{2} x + C}$$

c) Let $n = 4$

$$\int \cos^4 x dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x dx$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left(\frac{\sin 2x}{4} + \frac{1}{2} x \right) + C$$

$$= \boxed{\frac{1}{4} \cos^3 x \sin x + \frac{3}{16} \sin 2x + \frac{3}{8} x + C}$$

52.

$$\int x^n e^x dx$$

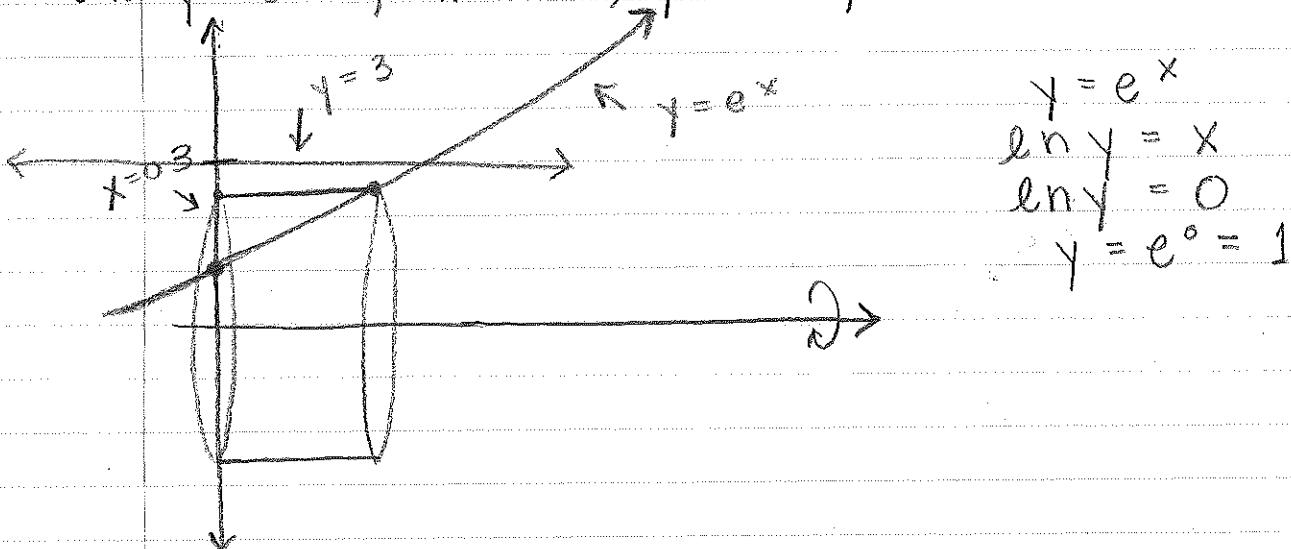
$$\text{Let } u = x^n \quad dv = e^x dx$$

$$du = nx^{n-1} dx \quad v = e^x$$

$$\int x^n e^x dx = x^n e^x - \int e^x n x^{n-1} dx$$

$$= x^n e^x - n \int e^x x^{n-1} dx$$

64. $y = e^x$, $x = 0$, $y = 3$; about x -axis



$$\begin{aligned}y &= e^x \\ \ln y &= x \\ \ln y &= 0 \\ y &= e^0 = 1\end{aligned}$$

radius = y , height = $\ln y$

$$V = \int_1^3 2\pi y \ln y dy = 2\pi \int_1^3 y \ln y dy$$

Let $u = \ln y$, $dv = y dy$

$$du = \frac{1}{y} dy, \quad v = \frac{1}{2} y^2$$

$$V = 2\pi \left[\frac{1}{2} y^2 \ln y \Big|_1^3 - \frac{1}{2} \int_1^3 \frac{1}{y} \cdot y^2 dy \right]$$

$$= 2\pi \left[\frac{1}{2} (3)^2 \ln(3) - \frac{1}{2} \ln(1) - \frac{1}{2} \int_0^3 y dy \right]$$

$$= 2\pi \left(\frac{9}{2} \ln(3) - \frac{1}{4} y^2 \Big|_1^3 \right)$$

$$= 2\pi \left(\frac{9}{2} \ln(3) - \frac{1}{4} [3^2 - 1^2] \right)$$

$$= 2\pi \left(\frac{9}{2} \ln(3) - 2 \right) = \boxed{\pi (9 \ln(3) - 4)}$$

$$70. \int_0^a f(x)g''(x)dx$$

$$\text{Let } u = f(x), \quad dv = g''(x)dx$$

$$du = f'(x)dx \quad v = g'(x)$$

$$\int_0^a f(x)g''(x)dx = \left[f(x)g'(x) \right]_0^a - \int_0^a f'(x)g'(x)dx$$

$$\text{Let } u = f'(x) \quad dv = g'(x)dx$$

$$du = f''(x)dx \quad v = g(x)$$

$$= f(a)g'(a) - f(0)g'(0) \rightarrow 0, f(0) = 0$$

$$- \left[f'(x)g(x) \right]_0^a - \int_0^a f''(x)g(x)dx$$

$$= f(a)g'(a) - \left[f'(a)g(a) - f'(0)g(0) \right] - \int_0^a f''(x)g(x)dx$$

$$= f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)dx.$$