

MATH 172 Homework 9

11.1 # 23, 24, 27, 72, 73, 75, 77, 74

23. $a_n = \frac{3+5n^2}{n+n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{\frac{3}{n^2} + 5}{\frac{n}{n^2} + 1}$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\frac{3}{n^2} + 5}{\frac{n}{n^2} + 1} = \frac{0+5}{0+1} = 5.$$

converges

24. $a_n = \frac{3+5n^2}{1+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{\frac{3}{n} + 5}{\frac{1}{n} + 1}$

$$\lim_{n \rightarrow \infty} \left(\frac{3}{n} + 5n \right) = \infty$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n} + 1 \right) = 0+1=1$$

$\Rightarrow a_n$ diverges

27. $a_n = 3^n 7^{-n} = \frac{3^n}{7^n} = \left(\frac{3}{7}\right)^n$

$$r = \frac{3}{7} < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{3}{7}\right)^n = 0$$

converges

72. $a_n = \cos n$

$$\{a_n\} = \{\cos n\} \approx \{0.54, -0.42, -0.99, -0.65, 0.28, \dots\}$$

The sequence increases & decreases.

\Rightarrow not monotonic

The sequence is bounded since

$$-1 \leq \cos n \leq 1 \text{ for all } n.$$

73. $a_n = \frac{1}{2n+3} \Rightarrow a_{n+1} = \frac{1}{2(n+1)+3} = \frac{1}{2n+5}$

$$\frac{1}{2n+5} = a_{n+1} < a_n = \frac{1}{2n+3}$$

$(a_n)_{n \geq 1}$ is strictly decreasing (monotonic)

The sequence is bounded since

a_n is always positive: $a_n > 0$ and decreases from $a_1 = \frac{1}{5} \geq a_n$ ($0 < a_n \leq \frac{1}{5}$)

$$74. a_n = \frac{1-n}{2+n} \Rightarrow a_{n+1} = \frac{1-(n+1)}{2+(n+1)} = \frac{-n}{3+n}$$

$\frac{1-n}{2+n} > \frac{-n}{3+n}$ To determine if this is true:

$$\Leftrightarrow (1-n)(3+n) > -n(2+n)$$

$$-n^2 - 2n + 3 > -n^2 - 2n$$

$$3 > 0$$

This is true for all $n \geq 1$.

$\Rightarrow (a_n)_{n=1}^{\infty}$ is strictly decreasing (monotonic)

Now, check if bounded:

$a_1 = 0$, since decreasing $a_n \leq 0$.

$$\lim_{n \rightarrow \infty} \frac{1-n}{2+n} = -1$$

$\Rightarrow -1 < a_n \leq 0$, bounded

$$75. a_n = n(-1)^n$$

The terms of a_n are alternating sign, so the sequence is not monotonic

$$\{a_n\} = \{-1, 2, -3, 4, -5, \dots\}$$

$$\lim_{n \rightarrow \infty} |a_n| = n = \infty,$$

the sequence is not bounded.

$$77. a_n = 3 - 2n e^{-n}$$

$$\text{Let } f(x) = 3 - 2x e^{-x}$$

$$f'(x) = -2(e^{-x} - xe^{-x}) = 2e^{-x}(x-1)$$

This is positive when $x > 1$.

$\Rightarrow f$ is increasing on $(1, \infty)$

This means that the sequence $(a_n)_{n=1}^{\infty} = \{f(n)\}_{n=1}^{\infty}$ is strictly increasing (monotonic)

The sequence is bounded below

$$\text{by } a_1 = 3 - 2(1)e^{-1} = 3 - 2e^{-1}$$

$$\lim_{n \rightarrow \infty} 3 - 2n e^{-n} = 3 - \lim_{n \rightarrow \infty} \frac{2n}{e^n} \xrightarrow[\text{Hopital's rule}]{} 3 - 0 = 3$$

$$3 - \lim_{n \rightarrow \infty} \frac{2}{e^n} = 3 - 0 = 3 \Rightarrow 3 - 2e^{-1} \leq a_n < 3 \quad (\text{bounded})$$