Calculus
MATH 172-Fall 2017
Lecture Notes

These notes are a concise summary of what has been covered so far during the lectures.

All the definitions must be memorized and understood. Statements of important theorems labelled in the margin with the symbol \& must be memorized and understood.

Proofs that you are expected to be able to reproduce and that are fundamental are labelled with the symbol in the margin.

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## CHAPTER 1

## Integration

## 1. The definite integral

If $\left(x_{n}\right)_{n=1}^{\infty}$ be a sequence of real numbers. We write $\sum_{i=1}^{n} x_{i}$ the sum of the $n$-th first term in the sequence, i.e.

$$
\sum_{i=1}^{n} x_{i}:=x_{1}+\cdots+x_{n}
$$

Definition 1 (Limit of a sequence). A sequence of real numbers $\left(x_{n}\right)_{n=1}^{\infty}$ is said to converge to a real number $\ell \in \mathbb{R}$, if for every $\varepsilon>0$ there exists a natural number $N:=N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N,\left|x_{n}-\ell\right|<\varepsilon$.

If $\left(x_{n}\right)_{n=1}^{\infty}$ converges to $\ell$, we write

$$
\lim _{n \rightarrow \infty} x_{n}=\ell
$$

Definition 2. [Integrable functions/Definite integral] Let $f$ be a function defined on a closed interval $[a, b]$ and let $\left(x_{i}\right)_{i=0}^{n}$ be the regular partition of $[a, b]$, i.e. $x_{i}=a+i \frac{b-a}{n}$ for all $i \in\{0,1, \ldots, n\}$. If $\lim _{n \rightarrow \infty} \sum_{i=1}^{n} f\left(x_{i}^{*}\right) \frac{b-a}{n}$ exists and gives the same value for all possible choices of points $x_{i}^{*} \in\left[x_{i-1}, x_{i}\right]$, then we say that $f$ is integrable on $[a, b]$.

If $f$ is integrable on $[a, b]$ we denote the definite integral of $f$ on $[a, b]$ by

$$
\int_{a}^{b} f(x) d x:=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} f\left(x_{i}^{*}\right) \frac{b-a}{n} .
$$

Example 1. Is the function $x \mapsto x^{2}+1$ integrable on $[0,1]$ ? If it is what is the value of $\int_{0}^{1}\left(x^{2}+1\right) d x$.

## 2. The Fundamental Theorem of Calculus

### 2.1. The Fundamental Theorem of Calculus I.

Theorem 1 (Fundamental Theorem of Calculus I). Let $f:[a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. Then, $F(x)=\int_{a}^{x} f(t) d t$ is well-defined for all $x \in[a, b]$. Moreover, the function $F$ is differentiable on $[a, b]$, and $F^{\prime}$ is continuous on $[a, b]$ with $F^{\prime}(x)=f(x)$ for all $x \in[a, b]$.

Sketch of proof.

Example 2. Describe where the following function is differentiable and compute its derivative.
(1) $F(x)=\int_{1}^{x} \cos (t) d t$

### 2.2. The Fundamental Theorem of Calculus II.

Definition 3. Let $f:[a, b] \rightarrow \mathbb{R}$. A function $F:[a, b] \rightarrow \mathbb{R}$ is said to be an anti-derivative of $f$ on $[a, b]$, if $F$ is differentiable on $[a, b]$ with $F^{\prime}(x)=f(x)$ for all $x \in[a, b]$.

Theorem 2 (Fundamental Theorem of Calculus II). Let $f:[a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, and let $F$ be an anti-derivative of $f$ on $[a, b]$. Then, $\int_{a}^{b} f(t) d t=$ $F(b)-F(a)$.

The FTC II combined with the Chain Rule has the following useful application.
Theorem 3. Let $I$ be an interval, $\varphi: I \rightarrow[a, b]$ be differentiable on $I$, and $f:[a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. Then, $H(x)=\int_{a}^{\varphi(x)} f(t) d t$ is well-defined for all $x \in I$, and the function $H$ is differentiable on $I$ with $H^{\prime}(x)=f(\varphi(x)) \varphi^{\prime}(x)$ for all $x \in I$.

Proof.

Example 3. Describe where the following function is differentiable and compute its derivative.
(1) $F(x)=\int_{1}^{x^{2}} t d t$

## 3. Change of variable or the substitution rule

Theorem 4 (Change of variable/Substitution rule). Let $\phi$ be a differentiable function on $[a, b]$ such that $\phi^{\prime}$ is continuous on $[a, b]$. Let $f$ be a continuous function on the range of $\phi$, i.e. on $\phi([a, b])$, then

$$
\int_{a}^{b} f(\phi(t)) \phi^{\prime}(t) d t=\int_{\phi(a)}^{\phi(b)} f(u) d u
$$

Exercise 1. Compute the following definite integrals:
(1) $\int_{2}^{3} \frac{3 x^{2}-1}{\left(x^{3}-x\right)^{2}} d x$
(2) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos (\theta)}{\sin ^{2}(\theta)} d \theta$
(3) $\int_{0}^{1} \frac{x}{1+x^{4}} d x$
(4) $\int_{2}^{e^{e}} \frac{d x}{x \ln (x)}$

The substitution rule is useful to compute anti-derivatives of complicated functions.

Exercise 2. Find all the anti-derivatives for the following functions:
(1) $f(x)=(2 x+1)\left(x^{2}+x+1\right)^{3}$
(2) $f(x)=\sqrt[3]{1-x}$
(3) $f(x)=\frac{x}{2 x^{2}+3}$
(4) $f(\theta)=\frac{\cos (\theta)}{1+\sin ^{2}(\theta)}$

## CHAPTER 2

## Applications of Integration I

## 1. Area between curves

In many cases, we can compute the area enclosed by certain curves by setting up a single integral.

The area enclosed by the curves $y=f(x), y=g(x), x=a$, and $x=b$ where $f, g$ are continuous functions on $[a, b]$ is

$$
\begin{equation*}
A=\int_{a}^{b}|f(x)-g(x)| d x \tag{1}
\end{equation*}
$$

The area enclosed by the curves $x=f(y), x=g(y), y=c$, and $y=d$, where $f, g$ are continuous functions on $[c, d]$ is

$$
\begin{equation*}
A=\int_{c}^{d}|f(y)-g(y)| d y \tag{2}
\end{equation*}
$$

Method: To solve a problem which asks to find the area of a region enclosed by several curves you need to:
(1) sketch the graph of the functions representing the curves in order to understand what is the region under consideration,
(2) determined what will be a judicious choice for the variable of integration,
(3) depending on how many times the curves interlace, set up one or several integrals that are needed in order to compute the area,
(4) find the intersection points of the curves in order to get the upper and lower bounds in the integral(s),
(5) compute the antiderivative(s) needed to compute the integral(s) using FTC II.

Exercise 3. Find the area of a region enclosed by the curves $y=x^{2}$ and $y=x^{4}$.

Solution.

Exercise 4. Find the area of a region enclosed by the curves $y=2 x+4$ and $4 x+y^{2}=0$.

Solution.

Exercise 5. Find the area of a region enclosed by the curves $x=3 y, x+y=0$ and $7 x+3 y=24$.

Solution.

## 2. Volumes

Computing the volume of a 3 -dimensional solid can be rather complicated and might require the use of a triple integral. However in certain situations, e.g. if the solid has certain symmetries, then we can compute the volume using a simple integral. The following formulas give the volume of solids of revolution obtained in by rotating a region of the $x y$-plane about one of the two axes.

The volume of the solid obtained by rotating about the $x$-axis the region bounded by the curves $y=f(x), y=g(x), x=a$, and $x=b$, where $f \geq g \geq 0$ are continuous functions on $[a, b]$ is

$$
\begin{equation*}
V=\pi \int_{a}^{b}\left(f(x)^{2}-g(x)^{2}\right) d x \tag{3}
\end{equation*}
$$

The volume of the solid obtained by rotating about the $y$-axis the region bounded by the curves $x=f(y), x=g(y), y=c$, and $y=d$, where $f \geq g \geq 0$ are continuous functions on $[c, d]$ is

$$
\begin{equation*}
V=\pi \int_{c}^{d}\left(f(y)^{2}-g(y)^{2}\right) d y \tag{4}
\end{equation*}
$$

For arbitrary solids, we can use the following formulas.

Let $S$ be a solid that lies between $x=a$ and $x=b$ and $A(x)$ be its crosssectional area in the plane passing through $x$ and perpendicular to the $x$-axis. If $x \mapsto A(x)$ is a continuous function on $[a, b]$, then the volume of $S$ is

$$
\begin{equation*}
V=\int_{a}^{b} A(x) d x \tag{5}
\end{equation*}
$$

Let $S$ be a solid that lies between $y=c$ and $y=d$ and $A(y)$ be its crosssectional area in the plane passing through $y$ and perpendicular to the $y$-axis. If $y \mapsto A(y)$ is a continuous function on $[c, d]$, then the volume of $S$ is

$$
\begin{equation*}
V=\int_{c}^{d} A(y) d y \tag{6}
\end{equation*}
$$

EXERCISE 6. Compute the volume of a frustum of a pyramid with height $h$, lower base side $a$, and top side $b \leq a$.

Solution.

## 3. Volumes by cylindrical shells

The volume of the solid obtained by rotating about the $y$-axis the region bounded by the curves $y=f(x), y=g(x), x=a$, and $x=b$, where $f$ and $g$ are continuous functions on $[a, b]$ with $0 \leq a<b$ is

$$
\begin{equation*}
V=2 \pi \int_{a}^{b} x|f(x)-g(x)| d x \tag{7}
\end{equation*}
$$

The volume of the solid obtained by rotating about the $x$-axis the region bounded by the curves $x=f(y), x=g(y), y=c$, and $y=d$, where $f$ and $g$ are continuous functions on $[c, d]$ with $0 \leq c<d$ is

$$
\begin{equation*}
V=2 \pi \int_{c}^{d} y|f(y)-g(y)| d y \tag{8}
\end{equation*}
$$

Exercise 7. Find the volume of the solid of revolution obtained by rotating the region enclosed by the curves $y=\sin \left(x^{2}\right), y=0, x=0, x=\sqrt{\pi}$, about the $y$-axis.

## Solution:

ExERCISE 8. Find the volume of the solid of revolution obtained by rotating the region enclosed by the curves $y=x^{2}-6 x+10, y=-x^{2}+6 x-6$, about the $y$-axis.

Solution:

Exercise 9. Find the volume of the solid of revolution obtained by rotating the region enclosed by the curves $y=x, x=0, x+y=2$, about the $x$-axis.

Solution:

Exercise 10. Find the volume of the solid of revolution obtained by rotating the region enclosed by the curves $y=x, x=0, x+y=2$, about the $y$-axis.

Solution:

Exercise 11. Find the volume of the solid of revolution obtained by rotating the region enclosed by the curves $y=x \sqrt{1+x^{3}}, y=0, x=0, x=2$, about the $y$-axis.

Solution:

Exercise 12. Find the volume of the solid of revolution obtained by rotating the region enclosed by the curves $x^{2}+(y-1)^{2}=1$, about the $y$-axis.

Solution:

## 4. Work

Definition 4. Suppose that an object moves along the $x$-axis in the positive from $x=a$ to $x=b$ and at each point between $a$ and $b$ a force $f(x)$ acts on the object, where $f$ is a continuous function on $[a, b]$. We define the work done in moving the object from $a$ to $b$ as

$$
W_{a \rightarrow b}=\int_{a}^{b} f(x) d x
$$

In particular if $f$ is a constant force, i.e. $f(x)=F$ for all $a \leq x \leq b$, then $W_{a \rightarrow b}=F(b-a)$.

Remark 1. The work done is an algebraic number and has to be interpreted accordingly.

## 5. Average value of a function

Definition 5. Let $a<b$ and $f$ be a continuous function on $[a, b]$. The average value of $f$ on $[a, b]$ is defined by

$$
f_{a v e}=\frac{1}{b-a} \int_{a}^{b} f(x) d x
$$

Recall the Mean Value Theorem.
Theorem 5 (Mean Value Theorem). Let $a<b$ and $f:[a, b] \rightarrow \mathbb{R}$. If,
(i) $f$ is continuous on $[a, b]$,
(ii) $f$ is differentiable on $(a, b)$,
then there exists $c \in(a, b)$ such that $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$.
We will not prove the Mean Value Theorem but we will use it to prove the Mean Value Theorem for Integrals which says that if $f$ is continuous on $[a, b]$ then there exists a real number $c \in[a, b]$ such that $f_{\text {ave }}=f(c)$.

Theorem 6 (Mean Value Theorem for Integrals). Let $f$ be a continuous function on $[a, b]$. Then, there exists a real number $c \in[a, b]$ such that

$$
\int_{a}^{b} f(x) d x=f(c)(b-a)
$$

Proof.

Exercise 13. The temperature of a metal rod, 10 m long, is $5 x\left(\right.$ in ${ }^{\circ} \mathrm{C}$ ) at a distance $x$ from one end of the rod. What is the average temperature of the rod?

Solution:

## CHAPTER 3

## Techniques of Integration

## 1. Integration by Parts

Theorem 7. Let $f, g$ be real valued functions on $[a, b]$ such that:
(1) $f$ and $g$ are differentiable on $[a, b]$,
(2) $f^{\prime}$ and $g^{\prime}$ are integrable on $[a, b]$.

Then,

$$
\begin{equation*}
\int_{a}^{b} f^{\prime}(x) g(x) d x=f(b) g(b)-f(a) g(a)-\int_{a}^{b} f(x) g^{\prime}(x) d x \tag{9}
\end{equation*}
$$

Proof. (Hint: Use the product rule and the FTC)

REMARK 2. It might be convenient in practice to write equation (9) as

$$
\begin{equation*}
\int_{a}^{b} f^{\prime}(x) g(x) d x=[f(x) g(x)]_{a}^{b}-\int_{a}^{b} f(x) g^{\prime}(x) d x \tag{10}
\end{equation*}
$$

Exercise 14. Let $f$ be a real-valued differentiable function on $[0,1]$ such that $f^{\prime}$ is integrable on $[0,1]$. If $f(0)=f(1)=0$, show that

$$
\int_{0}^{1} e^{x}\left(f(x)+f^{\prime}(x)\right) d x=0
$$

Solution:

The integration by part formula is extremely useful to compute antiderivative of functions of the form $P(t) e^{\alpha t}, P(t) \sin (\alpha t)$, or $P(t) \cos (\alpha t)$ where $P(t)$ is a polynomial. We typically apply the integration by part with $g=P$ in order to decrease the degree of the polynomial. If $P$ is a polynomial of degree $n$ we will apply the formula $n$-times in a row.

ExErcise 15. Compute the following integrals.
(1) $\int_{0}^{1} x^{2} e^{x}$.

Solution:
(2) $\int_{1}^{e^{\pi}} \sin (\ln (x))$.

Solution:
(3) $\int_{0}^{4} e^{\sqrt{x}} d x$.

Solution:

Exercise 16. Find all the antiderivatives of the function $f$, where:
(1) $f(x)=x \cos (x)$,

## Solution:

(2) $f(x)=\arcsin (x)$,

Solution:
(3) $f(x)=e^{-x} \cos (2 x)$,

Solution:
(4) $f(x)=x^{3} e^{x^{2}}$,

Solution:
(5) $f(x)=\cos ^{n}(x)$, where $n$ is a fixed natural number.

## Solution:

Exercise 17. Compute the volume of the solid of revolution obtained by rotating the region bounded by the curves $y=e^{x}, y=e^{-x}, x=1$, about the $y$-axis.

Solution:

## 2. Trigonometric Integrals

Our goal in this section is to compute antiderivatives of trigonometric functions.
We first consider functions of the form $f(x)=\sin (m x) \cos (n x), f(x)=\sin (m x) \sin (n x)$, $f(x)=\cos (m x) \cos (n x)$ where $m$ and $n$ are natural numbers.

If $f(x)=\sin (m x) \cos (n x)$ use the addition formula

$$
\sin (a) \cos (b)=\frac{1}{2}(\sin (a-b)+\sin (a+b)) .
$$

EXERCISE 18. Compute all the antiderivatives of the function $f(x)=\cos (3 x) \sin (x)$.
Solution:

If $f(x)=\sin (m x) \sin (n x)$ use the addition formula

$$
\sin (a) \sin (b)=\frac{1}{2}(\cos (a-b)-\cos (a+b)) .
$$

ExErcise 19. Compute all the antiderivative of the function $f(x)=\sin (3 x) \sin (x)$.
Solution:

If $f(x)=\cos (m x) \cos (n x)$ use the addition formula

$$
\cos (a) \cos (b)=\frac{1}{2}(\cos (a-b)+\cos (a+b)) .
$$

ExErcise 20. Compute all the antiderivative of the function $f(x)=\cos (3 x) \cos (x)$.
Solution:

We now consider functions of the form $f(x)=\sin ^{m}(x) \cos ^{n}(x)$ where $m$ and $n$ are natural numbers. We will use various changes of variables depending on the parity of $m$ or $n$.

If $f(x)=\sin ^{m}(x) \cos ^{n}(x)$ where $m$ is odd, i.e., if $f(x)=\sin ^{2 k+1}(x) \cos ^{n}(x)$ use the change of variable $u(x)=\cos (x)$.

ExERCISE 21. Compute all the antiderivatives of the function $f(x)=\sin ^{3}(x) \cos ^{2}(x)$.
Solution:

If $f(x)=\sin ^{m}(x) \cos ^{n}(x)$ where $n$ is odd, i.e., if $f(x)=\sin ^{m}(x) \cos ^{2 k+1}(x)$ use the change of variable $u(x)=\sin (x)$.

ExERCISE 22. Compute all the antiderivative of the function $f(x)=\cos ^{5}(x) \sin ^{2}(x)$.

## Solution:

EXERCISE 23. Compute all the antiderivative of the function $f(x)=x \cos ^{5}\left(x^{2}\right) \sin ^{2}\left(x^{2}\right)$.
Solution:

If $f(x)=\sin ^{m}(x) \cos ^{n}(x)$ where $m$ and $n$ are even, i.e., if $f(x)=$ $\sin ^{2 k}(x) \cos ^{2 r}(x)$, use the formulas
(1) $\sin ^{2}(x)=\frac{1-\cos (2 x)}{2}$,
(2) $\cos ^{2}(x)=\frac{1+\cos (2 x)}{2}$,
to progressively get rid of the exponents.
In the special case where $k=r$, you can also use $\sin (x) \cos (x)=\frac{\sin (2 x)}{2}$.

ExERCISE 24. Compute all the antiderivatives of the function $f(x)=\cos ^{2}(x) \sin ^{2}(x)$.

Solution:

Exercise 25. Compute $\int_{0}^{\frac{\pi}{2}} \cos ^{2}(x) \sin ^{2}(x) d x$.

Solution:

ExErcise 26. Compute the volume of the solid of revolution obtained by rotating the region bounded by the curves $y=\sin (x), y=2 \sin ^{2}(x), x=0, x=\frac{\pi}{2}$, about the $x$-axis.

## 3. Weierstrass Substitution

Computing all the antiderivatives of a trigonometric function can always be reduced to computing all the antiderivatives of a rational function using Weierstrass change of variable $t=\tan \left(\frac{\theta}{2}\right)$ for $-\pi<\theta<\pi$.

Let $t=\tan \left(\frac{\theta}{2}\right)$ for $-\pi<\theta<\pi$, then

- $\sin (\theta)=\frac{2 t}{1+t^{2}}$,
- $\cos (\theta)=\frac{1-t^{2}}{1+t^{2}}$,
- $\frac{d t}{d \theta}=\frac{1+t^{2}}{2}$.

How to compute all the antiderivatives of a rational function will be discussed later on.

Exercise 27. Find all the antiderivatives of $\frac{1}{\cos (\theta)}$.
Solution:

Exercise 28. Find all the antiderivatives of $\frac{1}{\sin (\theta)}$.
Solution:

## 4. Trigonometric Substitution

In this section we want to compute the antiderivative of functions involving expressions of the form $\sqrt{a^{2}-x^{2}}, \sqrt{a^{2}+x^{2}}, \sqrt{x^{2}-a^{2}}$.

In the case where you have an expression of the form $\sqrt{a^{2}-x^{2}}$ use the change of variable $x=a \sin (\theta)$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ or $x=a \cos (\theta)$ with $0 \leq \theta \leq \pi$.

Exercise 29. Compute $\int_{0}^{2} x^{3} \sqrt{4-x^{2}} d x$.
Solution:

Exercise 30. Find all the antiderivatives of $f(t)=\sqrt{9-e^{2 t}}$.
Solution:

In the case where you have an expression of the form $\sqrt{a^{2}+x^{2}}$ use the change of variable $x=a \tan (\theta)$ with $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

Exercise 31. Find all the antiderivatives of $f(x)=\frac{1}{\sqrt{x^{2}+1}}$.
Solution:

In the case where you have an expression of the form $\sqrt{x^{2}-a^{2}}$ use the change of variable $x=\frac{a}{\cos (\theta)}$ with $0 \leq \theta<\frac{\pi}{2}$ or $\pi \leq \theta<\frac{3 \pi}{2}$.

Exercise 32. Find all the antiderivatives of $f(x)=\frac{1}{x^{2} \sqrt{16 x^{2}-9}}$.

## Solution:

## 5. Integration of rational functions by partial fractions

It is possible to compute all the antiderivatives of rational functions, i.e., functions of the form $f(x)=\frac{P(x)}{Q(x)}$ where $P$ and $Q$ are polynomials.

Exercise 33. Let $a \neq 0$.
Find all the antiderivative of the function $f(x)=\frac{1}{a x+b}$
Solution:

ExErcise 34. Let $a \neq 0$.
Find all the antiderivative of the function $f(x)=\frac{2 a x+b}{a x^{2}+b x+c}$
Solution:

Exercise 35. Let $a \neq 0$.
Find all the antiderivative of the function $f(x)=\frac{1}{(a x+b)^{2}}$

## Solution:

Exercise 36. Let $\beta>0$.
Find all the antiderivative of the function $f(x)=\frac{1}{(x-\alpha)^{2}+\beta^{2}}$

## Solution

ExERCISE 37. Let $a \neq 0$.
Find all the antiderivative of the function $f(x)=\frac{1}{a x^{2}+b x+c}$ if $b^{2}-4 a c<0$.
Solution:

The general case can get rather tricky and we will restrict ourselves to functions of the form $f(x)=\frac{P(x)}{Q(x)}$ where $Q$ is a polynomial of degree at most 3 and the degree of $P$ is strictly less than the degree of $Q$.

### 5.1. Denominators with degree exactly 2 .

Two distinct roots: Let $f(x)=\frac{P(x)}{Q(x)}$ where $Q$ is a polynomial of degree exactly 2 and $P$ has degree at most 1 . To find all the antiderivatives of $f$, factorize $Q$ and if $f(x)=\frac{P(x)}{(x-a)(x-b)}$, with $a \neq b$, find $\alpha, \beta \in \mathbb{R}$ such that $f(x)=\frac{\alpha}{x-a}+\frac{\beta}{x-b}$.

EXERCISE 38. Compute all the antiderivatives of the function $f(x)=\frac{7}{2 x^{2}+5 x-12}$.

## Solution:

One repeated root: Let $f(x)=\frac{P(x)}{Q(x)}$ where $Q$ is a polynomial of degree exactly 2 and $P$ has degree at most 1 . To find all the antiderivatives of $f$, factorize $Q$ and if $f(x)=\frac{P(x)}{(x-a)^{2}}$, find $\alpha, \beta \in \mathbb{R}$ such that

$$
f(x)=\frac{\alpha}{x-a}+\frac{\beta}{(x-a)^{2}} .
$$

Exercise 39. Compute all the antiderivatives of the function $f(x)=\frac{3 x}{x^{2}+2 x+1}$.

## Solution:

Irreducible denominator: Let $f(x)=\frac{P(x)}{Q(x)}$ where $Q(x)=\alpha x+\beta$ and $Q(x)=x^{2}+b x+c$ is a polynomial of degree exactly 2 that does not have real roots and $P$ has degree at most 1 . To find all the antiderivatives of $f$, write $f(x)$ as $f(x)=\frac{\alpha}{2} \frac{2 x+b}{x^{2}+b x+c}+\left(\beta-\frac{\alpha b}{2}\right) \frac{1}{\left(x+\frac{b}{2}\right)^{2}+c-\frac{b^{2}}{4}}$ to be able to find the antiderivatives.

EXERCISE 40. Compute all the antiderivatives of the function $f(x)=\frac{2 x+6}{2 x^{2}+2 x+7}$.

## Solution:

### 5.2. Denominators with degree exactly 3 .

Three distinct roots: Let $f(x)=\frac{P(x)}{Q(x)}$ where $Q$ is a polynomial of degree exactly 3 and $P$ has degree at most 2 . To find all the antiderivatives of $f$, factorize $Q$ and if $f(x)=\frac{P(x)}{(x-a)(x-b)(x-c)}$, with $a, b, c$ all distinct, find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $f(x)=\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}$.

EXERCISE 41. Compute all the antiderivatives of the function $f(x)=\frac{1}{x^{3}+2 x^{2}-x+2}$.
Solution:

One simple root and a double root: Let $f(x)=\frac{P(x)}{Q(x)}$ where $Q$ is a polynomial of degree exactly 3 and $P$ has degree at most 2 . To find all the antiderivatives of $f$, factorize $Q$ and if $f(x)=\frac{P(x)}{(x-a)^{2}(x-b)}$, with $a \neq b$, find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $f(x)=\frac{\alpha}{x-a}+\frac{\beta}{(x-a)^{2}}+\frac{\gamma}{(x-b)}$.

ExERCISE 42. Compute all the antiderivatives of the function $f(x)=\frac{x^{2}}{2 x^{3}-x^{2}-45 x-12}$.

## Solution:

One triple root: Let $f(x)=\frac{P(x)}{Q(x)}$ where $Q$ is a polynomial of degree exactly 3 and $P$ has degree at most 2 . To find all the antiderivatives of $f$, factorize $Q$ and if $f(x)=\frac{P(x)}{(x-a)^{3}}$, find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $f(x)=\frac{\alpha}{x-a}+$ $\frac{\beta}{(x-a)^{2}}+\frac{\gamma}{(x-a)^{3}}$.

EXERCISE 43. Compute all the antiderivatives of the function $f(x)=\frac{2 x+1}{x^{3}-3 x^{2}+3 x-1}$.

## Solution:

One simple root and an irreducible factor: Let $f(x)=\frac{P(x)}{Q(x)}$ where $Q$ is a polynomial of degree exactly 3 . To find all the antiderivatives of $f$, factorize $Q$ and if $f(x)=\frac{P(x)}{(x-a)\left(x^{2}+b x+c\right)}$ where $x^{2}+b x+c$ does not have real roots, find $\alpha, \beta \in \mathbb{R}$ such that $f(x)=\frac{\alpha}{x-a}+\frac{\beta x+\gamma}{x^{2}+b x+c}$ and then write $f(x)$ in the following form $f(x)=\frac{\alpha}{x-a}+\frac{\beta}{2} \frac{2 x+b}{x^{2}+b x+c}+\left(\gamma-\frac{\beta b}{2}\right) \frac{1}{\left(x+\frac{b}{2}\right)^{2}+c-\frac{b^{2}}{4}}$ to
be able to find the antiderivatives.

ExErcise 44. Compute all the antiderivatives of the function $f(x)=\frac{1}{x^{3}-2 x^{2}+x-2}$. Solution:

## CHAPTER 4

## Improper Integrals

## 1. Integration on unbounded intervals

Definition 6 (Integrable functions on unbounded intervals).

- Let $f$ be a function defined on the interval $[a, \infty)$ such that for every $t \geq a$, $f$ is integrable on $[a, t]$. If $\lim _{t \rightarrow \infty} \int_{a}^{t} f(x) d x$ exists and is finite we say that $f$ is integrable on $[a, \infty)$ and we define the improper integral of $f$ on $[a, \infty)$ as

$$
\int_{a}^{\infty} f(x) d x:=\lim _{t \rightarrow \infty} \int_{a}^{t} f(x) d x
$$

- Let $f$ be a function defined on the interval $(-\infty, b]$ such that for every $t \leq b, f$ is integrable on $[t, b]$. If $\lim _{t \rightarrow-\infty} \int_{t}^{b} f(x) d x$ exists and is finite we say that $f$ is integrable on $(-\infty, b]$ and we define the improper integral of $f$ on $(-\infty, b]$ as

$$
\int_{-\infty}^{b} f(x) d x:=\lim _{t \rightarrow \infty} \int_{t}^{b} f(x) d x
$$

- Let $f$ be a function defined on the interval $(-\infty, \infty)$. If there exists $c \in(-\infty, \infty)$ such that $f$ is integrable on $(-\infty, c]$ and on $[c, \infty)$ then we say that $f$ is integrable on $(-\infty, \infty)$ and we define the improper integral of $f$ on $(-\infty, \infty)$ as

$$
\int_{-\infty}^{\infty} f(x) d x:=\lim _{t \rightarrow \infty} \int_{t}^{c} f(x) d x+\lim _{t \rightarrow \infty} \int_{c}^{t} f(x) d x
$$

REMARK 3. If the improper integral $\int_{a}^{\infty} f(x) d x\left(\right.$ or $\int_{-\infty}^{b} f(x) d x$ or $\left.\int_{-\infty}^{\infty} f(x) d x\right)$ exists and is finite we say that the improper integral is convergent. Otherwise we say that the improper integral is divergent.

Exercise 45. Is the function $f: x \mapsto \frac{1}{x}$ integrable on $[1, \infty)$ ? If it is, compute the improper integral of $f$ on $[1, \infty)$.

Solution:

ExERCISE 46. Is the function $f: x \mapsto \frac{1}{x^{2}}$ integrable on $[1, \infty)$ ? If it is, compute the improper integral of $f$ on $[1, \infty)$.

Solution:

ExErcise 47. For what values of $p$ is the function $f: x \mapsto \frac{1}{x^{p}}$ integrable on $[1, \infty)$ ? When it is, compute the improper integral of $f$ on $[1, \infty)$.

Solution:

Exercise 48. Is the function $f: \theta \mapsto \sin (\theta) e^{\cos (\theta)}$ integrable on $[0, \infty)$ ? If it is, compute the improper integral of $f$ on $[0, \infty)$.

Solution:

ExERCISE 49. For what values of $\alpha$ is the function $f: x \mapsto e^{\alpha x}$ integrable on $[0, \infty)$ ? When it is, compute the improper integral of $f$ on $[0, \infty)$.

## Solution:

## 2. Integrability of unbounded functions

Definition 7. [Improper integral of an unbounded functions on a bounded interval]

- Let $-\infty<a<b<\infty$ and $f$ be a function defined on the interval $[a, b)$ such that for every $a \leq t<b, f$ is integrable on $[a, t]$. If $\lim _{t \rightarrow b} \int_{a}^{t} f(x) d x$ exists and is finite we say that $f$ is integrable on $[a, b)$ and we define the improper integral of $f$ on $[a, b)$ as

$$
\int_{a}^{b} f(x) d x:=\lim _{t \rightarrow b^{-}} \int_{a}^{t} f(x) d x
$$

- Let $f$ be a function defined on the interval $(a, b]$ such that for every $a<$ $t \leq b, f$ is integrable on $[t, b]$. If $\lim _{t \rightarrow a} \int_{t}^{b} f(x) d x$ exists and is finite we say that $f$ is integrable on $(a, b]$ and we define the improper integral of $f$ on $(a, b]$ as

$$
\int_{a}^{b} f(x) d x:=\lim _{t \rightarrow a^{+}} \int_{t}^{b} f(x) d x .
$$

- Let $c \in(a, b)$ and $f$ be a function defined on the interval $[a, b]$ with an unbounded discontinuity at $c$. If $f$ is integrable on $[a, c)$ and on $(c, b]$ then we say that $f$ is integrable on $[a, b]$ and we define the improper integral of $f$ on $[a, b]$ as

$$
\int_{a}^{b} f(x) d x:=\int_{a}^{c} f(x) d x+\int_{c}^{b} f(x) d x
$$

REMARK 4. If the improper integral $\int_{a}^{b} f(x) d x$ exists and is finite we say that the improper integral is convergent. Otherwise we say that the improper integral is divergent.

Remark 5. A function that is integrable on $[a, b]$ is in particular defined on $[a, b)$ and integrable on $[a, t]$ for all $a \leq t<b$ and we need to make sure that the two definitions for $\int_{a}^{b} f(x) d x$ (Definition 2 and Definition 7 give the same number insuring the compatibility of the two definitions. This is indeed the case thanks to the FTC I. Assume that $f$ is a function that is integrable on $[a, b]$ then $F(t)=$ $\int_{a}^{t} f(x) d x$ is an anti-derivative of $f$. On the one hand, by the FTC I $F$ is continuous on $[a, b]$ and $\int_{a}^{b} f(x) d x=F(b)-F(a)=F(b)$ (according to Definition2 and the FTC II). On the other hand, by continuity of $F F(b)=\lim _{t \rightarrow b} F(t)=\lim _{t \rightarrow b} \int_{a}^{t} f(x) d x=$ $\int_{a}^{b} f(x) d x$ (according to Definition 7). Therefore both definitions are compatible.

Exercise 50. Is the function $f: x \mapsto \frac{1}{x}$ integrable on ( 0,1$]$ ? If it is, compute the improper integral of $f$ on $(0,1]$.

Solution:

Exercise 51. Is the function $f: x \mapsto \frac{1}{x^{2}}$ integrable on $(0,1]$ ? If it is, compute the improper integral of $f$ on $(0,1]$.

Solution:

Exercise 52. For what values of $p$ is the function $f: x \mapsto \frac{1}{x^{p}}$ integrable on $(0,1]$ ? When it is, compute the improper integral of $f$ on $(0,1]$.

Solution:

EXERCISE 53. Is the function $f: x \mapsto-\ln (x)$ integrable on $(0,1]$ ? If it is, compute the improper integral of $f$ on $(0,1]$.

## Solution

## 3. Comparison theorems

Theorem 8 (Comparison Theorem: convergence criterion). Let $f, g:[a, \infty) \rightarrow$ $\mathbb{R}$ such that
(1) $f$ and $g$ are continuous on $[a, \infty)$,
(2) $0 \leq f(x) \leq g(x)$ for all $x \geq a$,
(3) $g$ is integrable on $[a, \infty)$.

Then,
(1) $f$ is integrable on $[a, \infty)$ and
(2) $\int_{a}^{\infty} f(x) d x \leq \int_{a}^{\infty} g(x) d x$.

Remark 6. The conclusion of the convergence criterion is sometimes summarized as follows: If $\int_{a}^{\infty} g(x) d x$ is convergent, then $\int_{a}^{\infty} f(x) d x$ is convergent.

Exercise 54. Show that $\int_{0}^{\infty} e^{-x^{2}} d x$ is convergent.
Solution: [Hint: Split the integral at 1 and use the inequality $x^{2} \geq x$ for all $x \geq 1$ ]

Theorem 9 (Comparison Theorem: divergence criterion). Let $f, g:[a, \infty) \rightarrow$ $\mathbb{R}$ such that
(1) $f$ and $g$ are continuous on $[a, \infty)$,
(2) $0 \leq f(x) \leq g(x)$ for all $x \geq a$,
(3) $f$ is not integrable on $[a, \infty)$.

Then, $g$ is not integrable on $[a, \infty)$.

Remark 7. The conclusion of the divergence criterion is sometimes summarized as follows: If $\int_{a}^{\infty} f(x) d x$ is divergent, then $\int_{a}^{\infty} g(x) d x$ is divergent.

EXERCISE 55. Is the function $f: x \mapsto \frac{1+e^{-x}}{x}$ integrable on $[1, \infty)$ ?
Solution: [Hint: Use Exercise 47

Below is a list of useful inequalities.
(1) $x \leq x^{2}$ for all $x \geq 1$
(2) $x^{2} \leq x$ for all $0 \leq x \leq 1$
(3) $\sqrt{x} \leq x$ for all $x \geq 1$
(4) $x \leq \sqrt{x}$ for all $0 \leq x \leq 1$
(5) $\ln (x) \leq x$ for all $x \geq 0$
(6) $-1 \leq \cos (x) \leq 1$ for all $x \in \mathbb{R}$
(7) $-1 \leq \sin (x) \leq 1$ for all $x \in \mathbb{R}$
(8) $\sin (x) \leq x$ for all $x \geq 0$

## CHAPTER 5

## Sequences of Real Numbers

## 1. Definitions and basic properties

In Calculus, you are used to work with real-valued functions defined on the set (or a subset) of real numbers. A sequence is nothing else but a real-valued function whose domain is the set of natural numbers.

Definition 8. A sequence of real numbers is a function from $f: \mathbb{N} \rightarrow \mathbb{R}$.
The intuition and the tools to study sequences are completely different from the one to study arbitrary functions and we usually prefer to see a sequence not as a function defined on $\mathbb{N}$ but as a collection of real numbers that is enumerated in a special order. We will thus use a special notation instead of the functional notation, and a sequence whose values are $x_{n}:=f(n)$ will be simply denoted by $\left(x_{n}\right)_{n=1}^{\infty}$ or $\left(x_{n}\right)_{n \in \mathbb{N}}$ or $x_{1}, x_{2}, \ldots$ We say that $x_{n}$ is the term in the $n$-th position or the term of rank $n$. We might also say that $x_{n}$ is the generic term of the sequence.

Every function $g$ that is defined on $[1, \infty)$ can give rise to a sequence $\left(x_{n}\right)_{n=1}^{\infty}$ simply by defining $x_{n}=g(n)$ for all $n \geq 1$.

Definition 9. A sequence of real numbers $\left(x_{n}\right)_{n=1}^{\infty}$ is bounded above if there exists $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}, x_{n} \leq M$.

Definition 10. A sequence of real numbers $\left(x_{n}\right)_{n=1}^{\infty}$ is bounded below if there exists $m \in \mathbb{R}$ such that for all $n \in \mathbb{N}, x_{n} \geq m$.

Definition 11. A sequence of real numbers $\left(x_{n}\right)_{n=1}^{\infty}$ is bounded if there exist $m, M \in \mathbb{R}$ such that for all $n \in \mathbb{N}, m \leq x_{n} \leq M$.

Example 4. (1) The sequence defined by $x_{n}=n$ for all $n \in \mathbb{N}$ is not bounded above but is bounded below by $m=1$.
(2) The sequence defined by $x_{n}=2-\ln (n)$ for all $n \in \mathbb{N}$ is not bounded below but is bounded above by $M=2$.
(3) The sequence defined by $x_{n}=(-1)^{n}$ for all $n \in \mathbb{N}$ is bounded above by $M=1$ and bounded below by $m=-1$.
(4) The sequence defined by $x_{n}=\frac{1}{n}$ for all $n \in \mathbb{N}$ is bounded above by $M=1$ and bounded below by $m=0$.
(5) The sequence defined by $x_{n}=(-1)^{n} 2^{n}$ for all $n \in \mathbb{N}$ is neither bounded above nor bounded below.
(6)

Definition 12. A sequence $\left(x_{n}\right)_{n=1}^{\infty}$ is increasing if for all $n \in \mathbb{N} x_{n} \leq x_{n+1}$.
Definition 13. A sequence $\left(x_{n}\right)_{n=1}^{\infty}$ is strictly increasing if for all $n \in \mathbb{N}$ $x_{n}<x_{n+1}$.

Example 5. The sequence defined by $x_{n}=n$ for all $n \in \mathbb{N}$ is strictly increasing.
Definition 14. A sequence $\left(x_{n}\right)_{n=1}^{\infty}$ is decreasing if for all $n \in \mathbb{N} x_{n} \geq x_{n+1}$.
Definition 15. A sequence $\left(x_{n}\right)_{n=1}^{\infty}$ is strictly decreasing if for all $n \in \mathbb{N}$ $x_{n}>x_{n+1}$.

Example 6. The sequence defined by $x_{n}=2-3 n$ for all $n \in \mathbb{N}$ is strictly decreasing.

Definition 16. A sequence $\left(x_{n}\right)_{n=1}^{\infty}$ is monotonic (or monotone) if it is either increasing or decreasing.

Remark 8. There are sequences that are neither increasing nor decreasing, e.g. the sequence defined by $x_{n}=(-1)^{n}$ for all $n \in \mathbb{N}$.

To study the monotonicity of a sequence we usually study the sign of $x_{n+1}-x_{n}$.

- If $x_{n+1}-x_{n}>0$ for all $n \geq 1$ we can conclude that $\left(x_{n}\right)_{n=1}^{\infty}$ is strictly increasing.
- If $x_{n+1}-x_{n} \geq 0$ for all $n \geq 1$ we can conclude that $\left(x_{n}\right)_{n=1}^{\infty}$ is increasing.
- If $x_{n+1}-x_{n}<0$ for all $n \geq 1$ we can conclude that $\left(x_{n}\right)_{n=1}^{\infty}$ is strictly decreasing.
- If $x_{n+1}-x_{n} \leq 0$ for all $n \geq 1$ we can conclude that $\left(x_{n}\right)_{n=1}^{\infty}$ is decreasing.

For sequences whose terms are all positive (i.e. for all $n \geq 1, x_{n}>0$ ) we can look at the quotient $\frac{x_{n+1}}{x_{n}}$.

- If $\frac{x_{n+1}}{x_{n}}>1$ for all $n \geq 1$ we can conclude that $\left(x_{n}\right)_{n=1}^{\infty}$ is strictly increasing.
- If $\frac{x_{n+1}}{x_{n}} \geq 1$ for all $n \geq 1$ we can conclude that $\left(x_{n}\right)_{n=1}^{\infty}$ is increasing.
- If $\frac{x_{n+1}}{x_{n}}<1$ for all $n \geq 1$ we can conclude that $\left(x_{n}\right)_{n=1}^{\infty}$ is strictly decreasing.
- If $\frac{x_{n+1}}{x_{n}} \leq 1$ for all $n \geq 1$ we can conclude that $\left(x_{n}\right)_{n=1}^{\infty}$ is decreasing.

Exercise 56. Show that the sequence defined by $x_{n}=n^{2}$ for all $n \in \mathbb{N}$ is strictly increasing.

## Solution:

Exercise 57. Show that the sequence defined by $x_{n}=2^{-n}$ for all $n \in \mathbb{N}$ is strictly decreasing.

## Solution:

Exercise 58. Show that the sequence defined by $x_{n}=2-\ln (n)$ for all $n \in \mathbb{N}$ is strictly decreasing.

Solution:

Exercise 59. Show that the sequence defined by $x_{n}=\frac{1}{n}$ for all $n \in \mathbb{N}$ is strictly decreasing.

Solution:

Example 7. Consider the following recursively defined sequence:

$$
a_{n}= \begin{cases}1 & \text { if } n=1 \\ 3-\frac{1}{a_{n-1}} & \text { if } n \geq 2\end{cases}
$$

Can you show that the sequence $\left(a_{n}\right)_{n=1}^{\infty}$ is strictly increasing and bounded above by $M=3$ ? Is the sequence bounded below?

Definition 17. A sequence of real numbers $\left(x_{n}\right)_{n=1}^{\infty}$ is said to converge to a real number $\ell \in \mathbb{R}$, if for every $\varepsilon>0$ there exists $N:=N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N,\left|x_{n}-\ell\right|<\varepsilon$.

If a sequence $\left(x_{n}\right)_{n=1}^{\infty}$ converges to $\ell \in \mathbb{R}$, we write $\lim _{n \rightarrow \infty} x_{n}=\ell$
Remark 9. There is actually some freedom in the above definition. Indeed, one can replace $\left|x_{n}-\ell\right|<\varepsilon$ in the definition by $\left|x_{n}-\ell\right| \leq \varepsilon$ or $\left|x_{n}-\ell\right|<2 \varepsilon$ or $\left|x_{n}-\ell\right| \leq 100 \varepsilon$ for instance, and yet obtain an equivalent definition. This can be useful on occasion.

Proposition 1. A sequence of real numbers $\left(x_{n}\right)_{n=1}^{\infty}$ has at most one limit.

Example 8. (1) Let $c \in \mathbb{R}$ and consider the sequence defined by $x_{n}=c$ for all $n \in \mathbb{N}$. We can show that $\lim _{n \rightarrow \infty} x_{n}=c$.
(2) Consider the sequence defined by $x_{n}=(-1)^{n}$ for all $n \in \mathbb{N}$. We can show that $\left(x_{n}\right)_{n=1}^{\infty}$ is not convergent.
(3) Consider the sequence defined by $x_{n}=\frac{1}{n}$ for all $n \in \mathbb{N}$. We can show that $\lim _{n \rightarrow \infty} x_{n}=0$.

Theorem 10. Every convergent sequence is bounded.
Proof. Assume that $\left(x_{n}\right)_{n=1}^{\infty}$ converges to $\ell$. Then for $\varepsilon=1$ there exists $N \in \mathbb{N}$ such that for all $n \geq N,\left|x_{n}-\ell\right| \leq 1$, and thus $\ell-1<x_{n}<\ell+1$. Let $M:=\max \left\{x_{1}, x_{2}, \ldots, x_{N-1}, \ell+1\right\}$ and $m:=\min \left\{x_{1}, x_{2}, \ldots, x_{N-1}, \ell-1\right\}$, then for all $n \in \mathbb{N}, m \leq x_{n} \leq M$ and $\left(x_{n}\right)_{n=1}^{\infty}$ is bounded.

We will use repeatedly the following limit theorems.
Theorem 11. Let $\left(x_{n}\right)_{n=1}^{\infty}$ and $\left(y_{n}\right)_{n=1}^{\infty}$ be convergent sequences. Then, the sequence $\left(x_{n}+y_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty}\left(x_{n}+y_{n}\right)=\lim _{n \rightarrow \infty} x_{n}+\lim _{n \rightarrow \infty} y_{n}$.

Proof. Assume that $\lim _{n \rightarrow \infty} x_{n}=\ell_{1}<\infty$ and $\lim _{n \rightarrow \infty} y_{n}=\ell_{2}<\infty$. Let $\varepsilon>0$, then there exist $N_{1}, N_{2} \in \mathbb{N}$ such that for all $n \geq N_{1},\left|x_{n}-\ell_{1}\right|<\frac{\varepsilon}{2}$ and for all $n \geq N_{2},\left|y_{n}-\ell_{2}\right|<\frac{\varepsilon}{2}$. If follows from the triangle inequality that $\left|x_{n}+y_{n}-\left(\ell_{1}+\ell_{2}\right)\right|=\left|x_{n}-\ell_{1}+y_{n}-\ell_{2}\right| \leq\left|x_{n}-\ell_{1}\right|+\left|y_{n}-\ell_{2}\right|$, and hence for $n \geq N:=\max \left\{N_{1}, N_{2}\right\},\left|x_{n}+y_{n}-\left(\ell_{1}+\ell_{2}\right)\right| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.

Exercise 60. Let $\left(x_{n}\right)_{n=1}^{\infty}$ and $\left(y_{n}\right)_{n=1}^{\infty}$ be convergent sequences. Show that the sequence $\left(x_{n}-y_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty}\left(x_{n}-y_{n}\right)=\lim _{n \rightarrow \infty} x_{n}-$ $\lim _{n \rightarrow \infty} y_{n}$.

Solution:

ThEOREM 12. Let $\left(x_{n}\right)_{n=1}^{\infty}$ and $\left(y_{n}\right)_{n=1}^{\infty}$ be convergent sequences and $\lambda \in \mathbb{R}$. Then, the sequence $\left(\lambda x_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty}\left(\lambda x_{n}\right)=\lambda \lim _{n \rightarrow \infty} x_{n}$

Proof. Assume that $\lim _{n \rightarrow \infty} x_{n}=\ell_{1}<\infty$. If $\lambda=0$ the equality clearly holds. Otherwise, let $\varepsilon>0$, then there exist $N_{1} \in \mathbb{N}$ such that for all $n \geq N_{1}$, $\left|x_{n}-\ell_{1}\right|<\frac{\varepsilon}{\lambda}$ and simply remark that $\left|\lambda x_{n}-\lambda\right|=\lambda\left|x_{n}-\ell_{1}\right| \leq \lambda \frac{\varepsilon}{\lambda}=\varepsilon$.

Exercise 61. Let $\left(x_{n}\right)_{n=1}^{\infty}$ and $\left(y_{n}\right)_{n=1}^{\infty}$ be convergent sequences. Show that the sequence $\left(2 x_{n}+3 y_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty}\left(2 x_{n}+3 y_{n}\right)=2 \lim _{n \rightarrow \infty} x_{n}+$ $3 \lim _{n \rightarrow \infty} y_{n}$.

## Solution:

Theorem 13. Let $\left(x_{n}\right)_{n=1}^{\infty}$ and $\left(y_{n}\right)_{n=1}^{\infty}$ be convergent sequences. Then, the sequence $\left(x_{n} y_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty}\left(x_{n} \cdot y_{n}\right)=\lim _{n \rightarrow \infty} x_{n} \cdot \lim _{n \rightarrow \infty} y_{n}$.

Proof. Assume that $\lim _{n \rightarrow \infty} x_{n}=\ell_{1}<\infty$ and $\lim _{n \rightarrow \infty} y_{n}=\ell_{2}<\infty$. If follows from the triangle inequality that $\left|x_{n} \cdot y_{n}-\left(\ell_{1} \cdot \ell_{2}\right)\right|=\mid\left(x_{n}-\ell_{1}\right) y_{n}+\ell_{1}\left(y_{n}-\right.$ $\left.\ell_{2}\right)\left|\leq\left|x_{n}-\ell_{1}\right|\right| y_{n}\left|+\left|y_{n}-\ell_{2}\right|\right| \ell_{1} \mid$. Since $\left(y_{n}\right)_{n=1}^{\infty}$ is convergent, and thus bounded, there exists $M>0$ such that for all $n \in \mathbb{N},\left|y_{n}\right| \leq M$. Let $\varepsilon>0$. If $\left|\ell_{1}\right|>0$, then there exist $N_{1}, N_{2} \in \mathbb{N}$ such that for all $n \geq N_{1},\left|x_{n}-\ell_{1}\right|<\frac{\varepsilon}{2 M}$ and for all $n \geq N_{2},\left|y_{n}-\ell_{2}\right|<\frac{\varepsilon}{2\left|\ell_{1}\right|}$, and hence for $n \geq \max \left\{N_{1}, N_{2}\right\},\left|x_{n} \cdot y_{n}-\left(\ell_{1} \cdot \ell_{2}\right)\right|<$ $\frac{\varepsilon}{2 M} M+\frac{\varepsilon}{2\left|\ell_{1}\right|}\left|\ell_{1}\right|=\varepsilon$. If $\left|\ell_{1}\right|=0$ then for $n \geq \max \left\{N_{1}, N_{2}\right\},\left|x_{n} \cdot y_{n}\right|<\frac{\varepsilon}{2 M} M<\varepsilon$, and the proof is complete.

Theorem 14. Let $\left(x_{n}\right)_{n=1}^{\infty}$ be a convergent sequence. If $x_{n} \neq 0$ for all $n \in \mathbb{N}$ and $\lim _{n \rightarrow \infty} x_{n} \neq 0$, then the sequence $\left(\frac{1}{x_{n}}\right)_{n=1}^{\infty}$ is convergent and

$$
\lim _{n \rightarrow \infty} \frac{1}{x_{n}}=\frac{1}{\lim _{n \rightarrow \infty} x_{n}}
$$

Proof. Assume $\lim _{n \rightarrow \infty} x_{n}=\ell_{1} \neq 0$, then for $\varepsilon=\frac{\left|\ell_{1}\right|}{2}>0$ there exists $N_{1} \in \mathbb{N}$ such that for all $n \geq N_{1},\left|x_{n}-\ell_{1}\right|<\frac{\left|\ell_{1}\right|}{2}$. It follows from the reverse triangle inequality that $\left|x_{n}\right|>\frac{\left|\ell_{1}\right|}{2}>0$ for $n \geq N_{1}$. Also for $\varepsilon>0$ there exists $N_{2} \in \mathbb{N}$ such that for all $n \geq N_{2},\left|x_{n}-\ell_{1}\right|<\frac{\varepsilon\left|\ell_{1}\right|^{2}}{2}$ and for $n \geq \max \left\{N_{1}, N_{2}\right\}$, $\left|\frac{1}{x_{n}}-\frac{1}{\ell_{1}}\right|=\left|\frac{\ell_{1}-x_{n}}{x_{n} \ell_{1}}\right|<\frac{2}{\mid \ell_{1}}\left|\frac{x_{n}-\ell_{1}}{\ell_{1}}\right|<\varepsilon$.

THEOREM 15. Let $\left(x_{n}\right)_{n=1}^{\infty}$ and $\left(y_{n}\right)_{n=1}^{\infty}$ be convergent sequences. If $y_{n} \neq 0$ for all $n \in \mathbb{N}$ and $\lim _{n \rightarrow \infty} y_{n} \neq 0$, then the sequence $\left(\frac{x_{n}}{y_{n}}\right)_{n=1}^{\infty}$ is convergent and

$$
\lim _{n \rightarrow \infty} \frac{x_{n}}{y_{n}}=\frac{\lim _{n \rightarrow \infty} x_{n}}{\lim _{n \rightarrow \infty} y_{n}}
$$

Proof. Assume that $\lim _{n \rightarrow \infty} x_{n}=\ell_{1}<\infty$ and $\lim _{n \rightarrow \infty} y_{n}=\ell_{2} \neq 0$. Since $\frac{x_{n}}{y_{n}}=x_{n} \frac{1}{y_{n}}$, the result follows by combining Theorem 13 and Theorem 14

EXERCISE 62. If $\left(x_{n}\right)_{n=1}^{\infty}$ is convergent show that the sequence defined by $y_{n}=x_{n+1}$ for all $n \geq 1$ is convergent and that $\lim _{n \rightarrow \infty} y_{n}=\lim _{n \rightarrow \infty} x_{n}$, i.e., $\lim _{n \rightarrow \infty} x_{n+1}=\lim _{n \rightarrow \infty} x_{n}$.

## Solution:

## 2. Convergence Criteria

We will now try to find ways to recognize whether or not a sequence is convergent.

Theorem 16. Let $f:[0, \infty) \rightarrow \mathbb{R}$ be a function.

- If $\left(x_{n}\right)_{n=1}^{\infty}$ is the sequence such that $x_{n}=f(n)$ for all $n \in \mathbb{N}$
- if $\lim _{x \rightarrow \infty} f(x)=\ell$, then, the sequence $\left(x_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty} x_{n}=\ell$.

EXERCISE 63. Let $\left(x_{n}\right)_{n=1}^{\infty}$ be the sequence defined by $x_{n}=\frac{n^{2}}{2 n^{2}+n+5}$ for all $n \in \mathbb{N}$. Is the sequence $\left(x_{n}\right)_{n=1}^{\infty}$ convergent? If it is what is the limit of the sequence.

Solution:

ExErcise 64. Let $\left(x_{n}\right)_{n=1}^{\infty}$ be the sequence defined by $x_{n}=\frac{n^{3}}{3 n^{2}+2}$ for all $n \in \mathbb{N}$. Is the sequence $\left(x_{n}\right)_{n=1}^{\infty}$ convergent? If it is what is the limit of the sequence.

Solution:

Theorem 17 (The Squeeze Theorem). Let $\left(x_{n}\right)_{n=1}^{\infty},\left(y_{n}\right)_{n=1}^{\infty},\left(z_{n}\right)_{n=1}^{\infty}$ be sequences of real numbers. If

- there exists $n_{0} \in \mathbb{N}$ such that $x_{n} \leq y_{n} \leq z_{n}$ for all $n \geq n_{0}$,
- $\left(x_{n}\right)_{n=1}^{\infty}$ and $\left(z_{n}\right)_{n=1}^{\infty}$ are convergent and $\lim _{n \rightarrow \infty} x_{n}=\lim _{n \rightarrow \infty} z_{n}=\ell$, then, $\left(y_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty} y_{n}=\ell$.

Proof. Let $\varepsilon>0$, then there exist $N_{1}, N_{2} \in \mathbb{N}$ such that for all $n \geq \max \left\{N_{1}, N_{2}\right\}$, $\left|x_{n}-\ell\right|<\varepsilon$ and $\left|z_{n}-\ell\right|<\varepsilon$. Thus, if $n \geq \max \left\{N_{1}, N_{2}, n_{0}\right\}, \ell-\varepsilon<x_{n} \leq y_{n} \leq$ $z_{n}<\ell+\varepsilon$, and $-\varepsilon<y_{n}-\ell<\varepsilon$, which shows that $\left(y_{n}\right)_{n=1}^{\infty}$ is convergent to $\ell$.

Since $-\left|x_{n}\right| \leq x_{n} \leq\left|x_{n}\right|$ the following corollary follows from the Squeeze Theorem.

Corollary 1. Let $\left(x_{n}\right)_{n=1}^{\infty}$ be sequences of real numbers. If the sequence $\left(\left|x_{n}\right|\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty}\left|x_{n}\right|=0$ then the sequence $\left(x_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty} x_{n}=0$.

Example 9. Geometric sequences. Let $r \in \mathbb{R}$. We can show that the geometric sequence $\left(r^{n}\right)_{n=1}^{\infty}$ is:

- divergent if $r \leq-1$,
- convergent if $-1<r<1$ with $\lim _{n \rightarrow \infty} r^{n}=0$,
- convergent if $r=1$ with $\lim _{n \rightarrow \infty} r^{n}=1$,
- divergent if $r>1$ with $\lim _{n \rightarrow \infty} r^{n}=+\infty$.


## 3. The Monotone Convergence Theorem

The Monotone Convergent Theorem relates the convergence of a sequence with its monotonicity and boundedness.

THEOREM 18. Let $\left(x_{n}\right)_{n=1}^{\infty}$ be a sequence of real numbers. If $\left(x_{n}\right)_{n=1}^{\infty}$ is increasing and bounded above then $\left(x_{n}\right)_{n=1}^{\infty}$ is convergent.

THEOREM 19. Let $\left(x_{n}\right)_{n=1}^{\infty}$ be a sequence of real numbers. If $\left(x_{n}\right)_{n=1}^{\infty}$ is decreasing and bounded below then $\left(x_{n}\right)_{n=1}^{\infty}$ is convergent.

Hint. Use the approximation property of suprema to show that $\left(x_{n}\right)_{n=1}^{\infty}$ converges to $\sup \left\{x_{n}: n \in \mathbb{N}\right\}$ in (1) or to $\inf \left\{x_{n}: n \in \mathbb{N}\right\}$ in (2).

Combining Theorem 18 and Theorem 19 we obtain the Monotone Convergence Theorem also called the Monotonic Sequence Theorem.

Theorem 20. Every bounded monotone sequence is convergent.
Example 10. Consider the following recursively defined sequence:

$$
x_{n}= \begin{cases}1 & \text { if } n=1 \\ 3-\frac{1}{x_{n-1}} & \text { if } n \geq 2 .\end{cases}
$$

Is the sequence $\left(x_{n}\right)_{n=1}^{\infty}$ convergent? If it is what is its limit?

## Solution:

Example 11. Consider the following recursively defined sequence:

$$
x_{n}= \begin{cases}1 & \text { if } n=1 \\ \frac{1}{1+x_{n-1}} & \text { if } n \geq 2\end{cases}
$$

Assuming that the sequence $\left(x_{n}\right)_{n=1}^{\infty}$ is convergent and that $\lim _{n \rightarrow \infty} x_{n}=\ell \neq 0$, compute $\ell$.

Solution:

## CHAPTER 6

## Series of real numbers

## 1. Definitions, Notation, and Basic Properties

Definition 18. Let $\left(a_{n}\right)_{n=1}^{\infty}$ be a sequence of real numbers. For $n \geq 1$ we define the $n$-th partial sum, denoted $s_{n}$, as follows

$$
s_{n}=\sum_{k=1}^{n} a_{k}
$$

and we consider the sequence of partial sums $\left(s_{n}\right)_{n=1}^{\infty}$. If the sequence $\left(s_{n}\right)_{n=1}^{\infty}$ is convergent we say that the series with general term $a_{n}$ is convergent and if $\lim _{n \rightarrow \infty} s_{n}=\lim _{n \rightarrow \infty} \sum_{k=1}^{n} a_{k}=s$, then $s$ is called the sum of the series with general term $a_{n}$.

For convenience we will use the following notation. The series with general term $a_{n}$ is denoted by $\sum_{n} a_{n}$. If $\sum a_{n}$ is convergent and its sum is $s$ we write $\sum_{n=1}^{\infty} a_{n}=s$. In other words, $\sum_{n=1}^{\infty} a_{n}$ is a convenient notation for $\lim _{n \rightarrow \infty} \sum_{k=1}^{n} a_{k}$ whenever the limit exists and is finite.

Example 12. The series $\sum \frac{1}{n(n+2)}$ is convergent and its sum is

$$
\sum_{n=1}^{\infty} \frac{1}{n(n+2)}=\frac{3}{4}
$$

Solution:

$$
\begin{aligned}
s_{n}=\sum_{k=1}^{n} \frac{1}{k(k+2)} & =\sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right) \\
& =\frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k}-\sum_{k=1}^{n} \frac{1}{k+2}\right) \\
& =\frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k}-\sum_{i=3}^{n+2} \frac{1}{i}\right) \\
& =\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\sum_{k=3}^{n} \frac{1}{k}-\left(\sum_{i=3}^{n} \frac{1}{i}+\frac{1}{n+1}+\frac{1}{n+2}\right)\right) \\
& =\frac{1}{2}\left(1+\frac{1}{2}+\sum_{k=3}^{n} \frac{1}{k}-\sum_{i=3}^{n} \frac{1}{i}-\frac{1}{n+1}-\frac{1}{n+2}\right) \\
& =\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) \rightarrow_{n \rightarrow \infty} \frac{1}{2}\left(\frac{3}{2}-0-0\right)=\frac{3}{4} .
\end{aligned}
$$

Therefore, $\sum \frac{1}{n(n+2)}$ is convergent and $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}=\frac{3}{4}$.

Example 13. The series $\sum \ln \left(\frac{n}{n+1}\right)$ is not convergent. It actually diverges to $-\infty$.

Solution:

$$
\begin{aligned}
s_{n}=\sum_{k=1}^{n} \ln \left(\frac{k}{k+1}\right) & =\sum_{k=1}^{n}(\ln (k)-\ln (k+1)) \\
& =\sum_{k=1}^{n} \ln (k)-\sum_{k=1}^{n} \ln (k+1) \\
& =\sum_{k=1}^{n} \ln (k)-\sum_{i=2}^{n+1} \ln (i) \\
& =\ln (1)+\sum_{k=2}^{n} \ln (k)-\left(\sum_{i=2}^{n} \ln (i)+\ln (n+1)\right) \\
& =\ln (1)+\sum_{k=2}^{n} \ln (k)-\sum_{i=2}^{n} \ln (i)-\ln (n+1) \\
& =\ln (1)-\ln (n+1)=-\ln (n+1) \rightarrow_{n \rightarrow \infty}-\infty .
\end{aligned}
$$

Therefore, $\sum \ln \left(\frac{n}{n+1}\right)$ is not convergent since it diverges to $-\infty$.
Theorem 21 (Divergence Test). Let $\sum a_{n}$ be a series. If the series $\sum a_{n}$ is convergent then the sequence $\left(a_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty} a_{n}=0$.

The proof follows from the observation that $a_{n}=\sum_{k=1}^{n} a_{k}-\sum_{k=1}^{n-1} a_{k}$ for all $n \geq 2$.

Example 14. The series $\sum \frac{2 n-3}{n+5}$ is not convergent, since $\lim _{n \rightarrow \infty} \frac{2 n-3}{n+5}=2 \neq$ 0.

Theorem 22 (Limit theorems for series). Let $\sum a_{n}$ and $\sum b_{n}$ be series.
(1) If the series $\sum a_{n}$ and $\sum b_{n}$ are convergent then the series $\sum\left(a_{n}+b_{n}\right)$ is convergent and $\sum_{n=1}^{\infty}\left(a_{n}+b_{n}\right)=\sum_{n=1}^{\infty} a_{n}+\sum_{n=1}^{\infty} b_{n}$.
(2) Let $\lambda \in \mathbb{R}$. If the series $\sum a_{n}$ is convergent then the series $\sum\left(\lambda a_{n}\right)$ is convergent and $\sum_{n=1}^{\infty}\left(\lambda a_{n}\right)=\lambda \sum_{n=1}^{\infty} a_{n}$.

As we will see later analogue limit theorems for product and quotient of sequences do not hold for series.

Example 15 (Geometric series). For $r \in \mathbb{R}$, the series $\sum r^{n-1}$ is called the geometric series with ratio $r$. When $r=1$ the $n$-th partial sums are $s_{n}=n$ and the geometric series with ratio 1 is not convergent. For $r \neq 1$, we can also estimate the $n$-th partial sum $s_{n}=\sum_{k=1}^{n} r^{k-1}$ explicitly. Indeed,

$$
\begin{aligned}
r s_{n} & =r \sum_{k=1}^{n} r^{k-1}=\sum_{k=1}^{n} r^{k}=\sum_{k=0}^{n-1} r^{k}+r^{n}-r^{0} \\
& \left.=\sum_{i=1}^{n} r^{i-1}+r^{n}-1\right)(\text { change of index } i=k+1) \\
& =s_{n}+r^{n}-1
\end{aligned}
$$

And hence, $s_{n}(1-r)=1-r^{n}$ for all $n \geq 1$, and if $r \neq 1$

$$
s_{n}=\sum_{k=1}^{n} r^{k-1}=\frac{1-r^{n}}{1-r} .
$$

- If $r \leq-1,\left(s_{n}\right)_{n=1}^{\infty}$ is not convergent since does not have a limit.
- If $-1<r<1,\left(s_{n}\right)_{n=1}^{\infty}$ is convergent and $\lim _{n \rightarrow \infty} s_{n}=\frac{1}{1-r}$.
- If $r>1,\left(s_{n}\right)_{n=1}^{\infty}$ is divergent since $\lim _{n \rightarrow \infty} s_{n}=\infty$.

Therefore,

- If $r \leq-1$, the geometric series $\sum r^{n-1}$ is not convergent.
- If $-1<r<1$, the geometric series $\sum r^{n-1}$ is convergent and

$$
\sum_{n=1}^{\infty} r^{n-1}=\frac{1}{1-r}
$$

- If $r \geq 1$, the geometric series $\sum r^{n-1}$ is divergent towards $+\infty$.

ExERCISE 65. Is the series $\sum \frac{1}{n(n+1)}$ convergent (and if it is compute its sum)?

## Solution:

EXERCISE 66. Is the series $\sum \ln \left(\frac{n}{n+2}\right)$ convergent (and if it is compute its sum)? Solution:

Exercise 67. Is the series $\sum 3 r^{n-1}$ with $r \in(-1,1)$ convergent (and if it is compute its sum)?

Solution:

ExErcise 68. Is the series $\sum \frac{2^{n}}{3^{n+1}}$ convergent (and if it is compute its sum)? Solution:

ExErcise 69. Is the series $\sum \frac{4^{n+1}}{5^{n}}$ convergent (and if it is compute its sum)? Solution:

Exercise 70. Is the series $\sum \frac{5^{n+1}}{4^{n}}$ convergent (and if it is compute its sum)? Solution:

## 2. Comparison Theorems

Theorem 23 (Comparison theorems for series (convergence version)). Let $\sum a_{n}$ and $\sum b_{n}$ be series such that
(1) $0 \leq a_{n} \leq b_{n}$ for all $n \geq 1$,
(2) the series $\sum b_{n}$ is convergent.

Then, the series $\sum a_{n}$ is convergent and $\sum_{n=1}^{\infty} a_{n} \leq \sum_{n=1}^{\infty} b_{n}$.
REMARK 10. If we only assume that $0 \leq a_{n} \leq b_{n}$ for all $n \geq n_{0}$ for some $n_{0} \in \mathbb{N}$, then we can still conclude that the series $\sum a_{n}$ is convergent but it might not be true anymore that $\sum_{n=1}^{\infty} a_{n} \leq \sum_{n=1}^{\infty} b_{n}$.

Theorem 24 (Comparison theorems for series (divergence version)). Let $\sum a_{n}$ and $\sum b_{n}$ be series such that
(1) $0 \leq a_{n} \leq b_{n}$ for all $n \geq 1$,
(2) the series $\sum a_{n}$ is not convergent.

Then, the series $\sum b_{n}$ is not convergent.
Theorem 25 (Limit comparison theorem). Let $\sum a_{n}$ and $\sum b_{n}$ be series such that
(i) $a_{n}>0$ and $b_{n}>0$ for all $n \geq 1$,
(ii) there exists $c>0$ such that $\lim _{n \rightarrow \infty} \frac{a_{n}}{b_{n}}=c$,
then, $\sum a_{n}$ is convergent if and only if $\sum b_{n}$ is convergent.
Theorem 26 (Integral comparison theorem). Let $f:[1, \infty) \rightarrow[0, \infty)$. If
(i) $f$ is continuous on $[1, \infty)$,
(i) $f$ is decreasing on $[1, \infty)$. i.e. for all $x \leq y, f(x) \geq f(y)$.

Then,
(1) $\sum f(n)$ is convergent if and only if $f$ is integrable on $[1, \infty)$
and
(2) $\sum_{n=2}^{\infty} f(n) \leq \int_{1}^{\infty} f(x) d x \leq \sum_{n=1}^{\infty} f(n)$ whenever the limits exist.

In order to apply the integral comparison theorem to a series with nonnegative terms $\sum a_{n}$, the general term must be of the form $a_{n}=f(n)$ where $f$ is a (nonnegative) decreasing function.

Example 16. For what values of $s \in \mathbb{R}$ is the series $\sum \frac{1}{n^{s}}$ convergent?
Solution:

EXERCISE 71. Is the series $\sum \frac{1}{(n+1) \ln (n+1)}$ convergent (and if it is determine its sum)?

## Solution:

## 3. Alternating Series

We say that a sequence $\left(b_{n}\right)_{n=1}^{\infty}$ has constant sign if either $b_{n} \geq 0$ for all $n \geq 1$ or $b_{n} \leq 0$ for all $n \geq 1$.

Definition 19. A series $\sum a_{n}$ is an alternating series if for all $n \geq 1, a_{n}=$ $(-1)^{n} b_{n}$ where $\left(b_{n}\right)_{n=1}^{\infty}$ is a sequence of constant sign.

Example 17. The alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$ is an alternating series since $\frac{(-1)^{n-1}}{n}=(-1)^{n}\left(\frac{-1}{n}\right)$, and $\frac{-1}{n} \leq 0$ for all $n \geq 1$.

ThEOREM 27 (Alternating series theorem). Let $\sum a_{n}$ be an alternating series. If
(1) $\lim _{n \rightarrow \infty}\left|a_{n}\right|=0$, and
(2) $\left(\left|a_{n}\right|\right)_{n=1}^{\infty}$ is decreasing, then $\sum a_{n}$ is convergent.

Exercise 72. Using the alternating series theorem, show that the alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$ is convergent.

Solution:

## 4. Absolutely Convergent Series

Definition 20. A series $\sum a_{n}$ is absolutely convergent if the series $\sum\left|a_{n}\right|$ is convergent.

Remark 11. A convergent series is not necessarily absolutely convergent. For instance, the alternating harmonic series is convergent but not absolutely convergent.

ThEOREM 28 (Absolute convergence implies convergence). If a series $\sum a_{n}$ is absolutely convergent then $\sum a_{n}$ is convergent.

Theorem 29 (The ratio test). Let $\sum a_{n}$ be a series.
(1) if $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=L>1$ then $\sum a_{n}$ is not convergent.
(2) if $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=L<1$ then $\sum a_{n}$ is absolutely convergent (and in particular convergent).

REmark 12. If $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=1$ then the ratio test is inconclusive. For instance, $\sum \frac{1}{n}$ is not convergent, $\sum \frac{(-1)^{n}}{n}$ is convergent but not absolutely convergent, and $\sum \frac{1}{n^{2}}$ is absolutely convergent. For all these series the limit of the ratio is 1 .

EXERCISE 73. Is the series $\sum \frac{(-3)^{n-1}}{\sqrt{n}}$ convergent?
Solution:

## CHAPTER 7

## Applications of Integration II

## 1. Arc length

If $f:[a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ and $f^{\prime}$ is continuous on $[a, b]$, the length of the arc of the curve represented by $y=f(x)$ for $a \leq x \leq b$ is

$$
\begin{equation*}
L=\int_{a}^{b} \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x \tag{11}
\end{equation*}
$$

EXERCISE 74. Compute the length of the curve represented by $y=\frac{x^{3}}{6}+\frac{1}{2 x}$ for $1 \leq x \leq 2$.

## Solution:

$$
L=\frac{17}{12} .
$$

If $g:[c, d] \rightarrow \mathbb{R}$ is differentiable on $[c, d]$ and $g^{\prime}$ is continuous on $[c, d]$, the length of the curve represented by $x=g(y)$ for $c \leq y \leq d$ is

$$
\begin{equation*}
L=\int_{c}^{d} \sqrt{1+\left(g^{\prime}(y)\right)^{2}} d y \tag{12}
\end{equation*}
$$

Exercise 75. Compute the length of the curve represented by $x=y^{3 / 2}$ for $0 \leq y \leq 1$.

Solution:
$L=\frac{13 \sqrt{13}-8}{27}$.

## 2. Area of a surface of revolution

If $f:[a . b] \rightarrow[0, \infty)$ is differentiable on $[a, b]$ and $f^{\prime}$ is continuous on $[a, b]$, the surface area obtained by rotating about the $x$-axis the curve $y=f(x)$ for $a \leq x \leq b$ is

$$
\begin{equation*}
S=2 \pi \int_{a}^{b} f(x) \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x \tag{13}
\end{equation*}
$$

Exercise 76. Compute the surface area obtained by rotating about the $x$-axis the curve $y=\cos (x)$ for $0 \leq x \leq \frac{\pi}{3}$.

## Solution:

If $g:[c, d] \rightarrow[0, \infty)$ is differentiable on $[c, d]$ and $g^{\prime}$ is continuous on $[c, d]$, the surface area obtained by rotating about the $y$-axis the curve $x=g(y)$ for $c \leq y \leq d$ is

$$
\begin{equation*}
S=2 \pi \int_{c}^{d} g(y) \sqrt{1+\left(g^{\prime}(y)\right)^{2}} d y \tag{14}
\end{equation*}
$$

ExERCISE 77. Compute the surface area obtained by rotating about the $y$-axis the curve $y^{2}=4 x+4$ for $0 \leq x \leq 8$.

Solution:

$$
S=\frac{8 \pi}{3}(10 \sqrt{10}-2 \sqrt{2}) .
$$

## CHAPTER 8

## Parametric equations and polar equations

Some curves can be described conveniently using various equations. Considering a circle is a simple but enlightening example.

The circle of radius $R>0$ centered at the origin of the plan can be represented as:

- the curve represented by the Cartesian equation: $x^{2}+y^{2}=R^{2}$
- the curve that is the combination of the graph of two functions: $y=$ $\sqrt{R^{2}-x^{2}}$ and $y=-\sqrt{R^{2}-x^{2}}$ for $-R \leq x \leq R$.
- the parametric curve:

$$
\left\{\begin{array}{l}
x(t)=R \cos (t), t \in[0,2 \pi] \\
y(t)=R \sin t
\end{array}\right.
$$

- the parametric curve:

$$
\left\{\begin{array}{l}
x(t)=R \sin (2 t), t \in[0, \pi] \\
y(t)=R \cos 2 t
\end{array}\right.
$$

- the polar curve: $r(\theta)=R$ for $\theta \in[0,2 \pi]$.


## 1. Parametric curves

Let $f, g: I \rightarrow \mathbb{R}$ be functions defined on an interval $I$. The collection of points $M(t)=(f(t), g(t))$ when $t \in I$ describes a curve of the 2 -dimensional space, called a parametric curve. A representation of the parametric curve is given by the system of two equations :

$$
\left\{\begin{array}{l}
x=f(t), t \in I \\
y=g(t)
\end{array}\right.
$$

where the variable $t$ is called the parameter. The support of the curve is the set of points $M(t)=(f(t), g(t))$ when $t \in I$. Parametric curves are useful to describe complicated curves (e.g. those who are backtracking) and trajectories of objects where the parameter represents time.

Every function $y=f(x)$ (resp. $x=g(y)$ ) can be turned into a parametric equation by setting either $x=t$ (resp. $y=t$ ).

Example 18.
ExERCISE 78. Eliminate the parameter to find a Cartesian equation of the curve

$$
\left\{\begin{array}{l}
x=3 t+1, t \in \mathbb{R} \\
y=5 t-2
\end{array}\right.
$$

Exercise 79. Eliminate the parameter to find a Cartesian equation of the curve

$$
\left\{\begin{array}{l}
x=\frac{\cos (\theta)}{3}, 0 \leq \theta \leq \pi \\
y=5 \sin (\theta)
\end{array}\right.
$$

### 1.1. Arc length of parametric curves.

Consider a parametric curve

$$
\left\{\begin{array}{l}
x=f(t), t \in[a, b] \\
y=g(t)
\end{array}\right.
$$

such that $f$ and $g$ are differentiable on $[a, b]$, and $f^{\prime}$ and $g^{\prime}$ are continuous and not simultaneously 0 on $(a, b)$. If the curve is traversed once as $t$ increases from $a$ to $b$, then the arc length of the parametric curve is

$$
\begin{equation*}
L=\int_{a}^{b} \sqrt{\left(\left(f^{\prime}(t)\right)^{2}+\left(g^{\prime}(t)\right)^{2}\right.} d t=\int_{a}^{b} \sqrt{\left(\frac{d x}{d t}\right)^{2}+\left(\frac{d y}{d t}\right)^{2}} d t . \tag{15}
\end{equation*}
$$

Exercise 80. Find the length of the curve defined by

$$
\left\{\begin{array}{l}
x=1+3 t^{2}, t \in[0,1] \\
y=4+2 t^{3}
\end{array}\right.
$$

Solution:

### 1.2. Area enclosed by a parametric curve.

The area of the region between a parametric curve

$$
\left\{\begin{array}{l}
x=f(t), t \in[a, b] \\
y=g(t)
\end{array}\right.
$$

that is traced out once for $t \in[a, b]$ where $g \geq 0$ and the $x$-axis is

$$
\begin{equation*}
A=\int_{a}^{b} g(t) f^{\prime}(t) d t=\int_{a}^{b} y \frac{d x}{d t} d t, \text { if } f^{\prime} \geq 0 \tag{16}
\end{equation*}
$$

$$
\begin{equation*}
A=-\int_{a}^{b} g(t) f^{\prime}(t) d t=-\int_{a}^{b} y \frac{d x}{d t} d t, \text { if } f^{\prime} \leq 0 \tag{17}
\end{equation*}
$$

Exercise 81. Find the area under the arch of the cycloid

$$
\left\{\begin{array}{l}
x=r(\theta-\sin (\theta)) \\
y=r(1-\cos (\theta))
\end{array}\right.
$$

for $\theta \in[0,2 \pi]$.

## Solution:

$$
A=3 \pi r^{2}
$$

## 2. Polar coordinates

In the Cartesian coordinate system the coordinates of a point are given in the form $(x, y)$. The polar coordinate system is defined by fixing an origin, the pole, and a ray emanating from the pole, the polar axis. Typically, the pole is taken to be the origin of the 2-dimensional plane and the polar axis is a horizontal ray pointing towards the right. The polar coordinates of a point is given in the form $(r, \theta)$ where $\theta$ is an angle in radian and $r$ a real number. If $r \geq 0$, the point with polar coordinates $(r, \theta)$ is located at a distance $r$ on the ray that is rotated counterclockwise of an angle $\theta$ starting from the polar axis. If $r \leq 0$, we adopt the convention that the point with polar coordinates $(r, \theta)$ is the point with polar coordinates $(-r, \theta+\pi)$. If a point has polar coordinates $(r, \theta)$ its Cartesian coordinates are $(r \cos (\theta), r \sin (\theta))$. If a point has Cartesian coordinates $(x, y)$ with $x>0$ and $y \geq 0$ then it polar coordinates are $\left(\sqrt{x^{2}+y^{2}}, \arctan \left(\frac{y}{x}\right)\right)$ (the other cases are treated in the book page ).

### 2.1. Arc length in polar coordinates.

Let $r=f(\theta)$ be a polar equation of a curve between the rays $\theta=a$ and $\theta=b$ such that $f$ is differentiable on $[a, b]$, and $f^{\prime}$ is continuous on $[a, b]$. The arc length of the polar curve $r=f(\theta)$ between the rays $\theta=a$ and $\theta=b$ is

$$
\begin{equation*}
L=\int_{a}^{b} \sqrt{f(\theta)^{2}+\left(f^{\prime}(\theta)\right)^{2}} d \theta=\int_{a}^{b} \sqrt{r^{2}+\left(\frac{d r}{d \theta}\right)^{2}} d \theta \tag{18}
\end{equation*}
$$

Exercise 82. Find the length of the cardioid whose polar equation is $r=$ $1+\sin (\theta)$.

## $L=8$

### 2.2. Area in polar coordinates.

The area of the region enclosed by the polar curve $r=f(\theta)$ and the rays $\theta=a$ and $\theta=b$, where $f$ is a positive continuous function and where $0<b-a \leq 2 \pi$ is

$$
\begin{equation*}
A=\frac{1}{2} \int_{a}^{b} f(\theta)^{2} d \theta=\frac{1}{2} \int_{a}^{b} r^{2} d \theta \tag{19}
\end{equation*}
$$

Exercise 83. Find the area of the region that is enclosed by the curve $r=e^{\frac{\theta}{4}}$ and that lies in the sector $\frac{\pi}{3} \leq \theta \leq \pi$.

Solution:

$$
A=e^{\frac{\pi}{2}}-e^{\frac{\pi}{3}}
$$

## CHAPTER 9

## Power Series

We have considered numerical series $\sum a_{n}$ where $\left(a_{n}\right)_{n=1}^{\infty}$ is a sequence of real numbers. If we were given infinitely many functions $f_{n}: \mathbb{R} \rightarrow \mathbb{R}, n \geq 1$, we could define for every $x$ in the domain of definition of all the functions $f_{n}$ the sequence $\left(f_{n}(x)\right)_{n=1}^{\infty}$, and then the numerical series $\sum f_{n}(x)$. Depending on the values taken by the variable $x$ the series $\sum f_{n}(x)$ might or might not converge.

Definition 21 (Domain of definition). For every $n \geq 1$, let $f_{n}$ be a real-valued function. The domain of definition of a series of the form $\sum f_{n}(x)$ consists of all the values of $x$ for which the power series converge.

Geometric series can fruitfully be seen as numerical series constructed from the above process. Indeed, if one considers the functions $f_{n}(x)=x^{n-1}$ for $n \geq 1$ (with the convention that $x^{0}=1$ ) then the series $\sum f_{n}(x)$ is nothing else but the geometric series $\sum x^{n-1}$ which is convergent if and only if $|x|<1$. And thus for every $x \in(-1,1)$ we can define a new function $f(x)=\sum_{n=1}^{\infty} x^{n-1}=\sum_{n=0}^{\infty} x^{n}$. Previous computations show that $f(x)=\frac{1}{1-x}$. A power series (centered at 0 ) in the real variable $x$ is a series of the form $\sum f_{n}(x)$ where $f_{n}$ is a power function of degree $n$, i.e., $f_{n}(x)=c_{n} x^{n}$ where $c_{n}$ is a real number. If not specified, the sum of a power series starts at $n=0$.

Definition 22. A power series (in the real variable $x$ ) centered at $a \in \mathbb{R}$ is a series of the form $\sum_{n=0}^{\infty} c_{n}(x-a)^{n}$ where $\left(c_{n}\right)_{n=1}^{\infty}$ is a sequence of real numbers.

We will discuss three main questions about power series:

- What is the domain of definition of a power series?
- What are the differentiation and integration rules for power series?
- What functions admit a power series representation?


## 1. Radius of convergence

The particular form of the general term of a power series has a strong influence on its domain of definition. Note first that the domain of definition of a power series centered at $a$ is non-empty since the series is convergent at least for $x=a$ (in this case at most one term is non-zero and we have a finite sum).

THEOREM 30. Let $\sum_{n=0}^{\infty} c_{n}(x-a)^{n}$ be a power series centered at $a$. The following trichotomy holds:
(1) The power series converges only when $x=a$.
(2) The power series converges for all $x \in \mathbb{R}$.
(3) There exists a (positive) number $R>0$ such that the power series converges if $|x-a|<R$ and does not converge if $|x-a|>R$.

Definition 23. The number $R$ such that the third assertion in the trichotomy holds is called the radius of convergence of the power series centered at $a$. By convention we write $R=0$ if the first assertion holds and $R=\infty$ if the second assertion holds.

REmARK 13. At the borderline case when $x=R$ everything can happen. The domain of definition of a power series is an interval (called the interval of convergence) of one of the following forms:

- $\{a\}$,
- $(a-R, a+R)$,
- $[a-R, a+R)$,
- $(a-R, a+R]$,
- $[a-R, a+R]$,
- $(-\infty,+\infty)$.

Example 19. The radius of convergence of the power series $\sum_{n=0}^{\infty}(x-a)^{n}$ is $R=1$ and its interval of convergence is $(a-1, a+1)$.

Example 20. The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^{n}}{n}$ is $R=1$ and its interval of convergence is $[-1,1)$.

Example 21. The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n} x^{n}}{n}$ is $R=1$ and its interval of convergence is $(-1,1]$.

Example 22. The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^{n}}{n^{2}}$ is $R=1$ and its interval of convergence is $[-1,1]$.

Example 23. The radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{n}}{n!}$ is $R=\infty$ and its interval of convergence is $(-\infty,+\infty)$.

To determine the interval of convergence of a power series centered at $a$ you need to
(1) determine the radius of convergence (usually using the ratio test),
(2) determine the behavior of the power series at the two endpoints $a-R$ and $a+R$ (in the case where the radius of convergence is positive and finite).
EXERCISE 84. Determine the interval of convergence of the power series

$$
\sum_{n=0}^{\infty} n!(2 x-1)^{n}
$$

Solution:

ExERCISE 85. Determine the interval of convergence of the power series

$$
\sum_{n=1}^{\infty} \frac{(x-2)^{n}}{n}
$$

Solution:

EXERCISE 86. Determine the interval of convergence of the power series

$$
\sum_{n=1}^{\infty} \frac{(2 x+3)^{n}}{n}
$$

Solution:

ExErcise 87. Determine the interval of convergence of the power series

$$
\sum_{n=2}^{\infty} \frac{(x+1)^{n}}{n \ln (n)}
$$

Solution:

Exercise 88. Determine the interval of convergence of the power series

$$
\sum_{n=1}^{\infty} \frac{(4 x+1)^{n}}{n^{2}}
$$

## Solution:

## 2. Differentiation and integration properties of power series

Theorem 31 (Term-by-term integration). If the power series $\sum_{n=0}^{\infty} c_{n}(x-a)^{n}$ has convergence radius $R$, then the function $f:(a-R, a+R) \rightarrow \mathbb{R}$, defined by $f(x)=\sum_{n=0}^{\infty} c_{n}(x-a)^{n}$ is continuous on $(a-R, a+R)$ and the anti-derivatives of $f$ are of the form $F(x)=\sum_{n=0}^{\infty} c_{n} \frac{(x-a)^{n+1}}{n+1}+C$ for some constant $C \in \mathbb{R}$. Moreover the radius of convergence of the integrated series is also $R$.

Example 24. The power series $\sum_{n=0}^{\infty} x^{n}$ is continuous on $(-1,+1)$ and its antiderivatives are the functions defined by $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}+C$ for some constant $C \in \mathbb{R}$. Using the ratio test we can verify that the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ is also $R=1$.

Remark 14. Recall that $\sum_{n=0}^{\infty} x^{n}=\frac{1}{1-x}$ for all $x \in(-1,1)$, and then it follows that the anti-derivative of $\frac{1}{1-x}$ that is 0 when $x=0$ is the function defined by $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ for all $x \in(-1,1)$. Since we already know that the anti-derivative of $\frac{1}{1-x}$ that is 0 when $x=0$ is $-\ln |1-x|$ we have $\ln (1-x)=-\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ for all $x \in(-1,1)$ and thus $\ln (x)=\sum_{n=0}^{\infty} \frac{(-1)^{n}(x-1)^{n+1}}{n+1}$ whenever $|x-1|<1$.

Exercise 89. Let $f:(-1,1) \rightarrow \mathbb{R}$ be the function defined by $f(x)=\frac{x}{1+x^{5}}$. Determine all the antiderivatives of $f$.

ExERCISE 90. Let $f:(-1,1) \rightarrow \mathbb{R}$ be the function defined by $f(x)=\frac{1-x^{2}}{1+x^{3}}$. Determine all the antiderivatives of $f$.

## Solution:

Theorem 32 (Term-by-term differentiation). If the power series $\sum_{n=0}^{\infty} c_{n}(x-$ $a)^{n}$ has convergence radius $R$, then the function $f:(a-R, a+R) \rightarrow \mathbb{R}$, defined by $f(x)=\sum_{n=0}^{\infty} c_{n}(x-a)^{n}$ is differentiable on $(a-R, a+R)$ and $f^{\prime}(x)=\sum_{n=1}^{\infty} n c_{n}(x-$ $a)^{n-1}$. Moreover the radius of convergence of the derived series is also $R$.

Example 25. The power series $\sum_{n=0}^{\infty} x^{n}$ is differentiable on $(-1,+1)$ and its derivative is the function defined by $\sum_{n=1}^{\infty} n x^{n-1}$. Using the ratio test we can verify that the radius of convergence of the power series $\sum_{n=1}^{\infty} n x^{n-1}$ is also $R=1$.

Remark 15. Recall that $\sum_{n=0}^{\infty} x^{n}=\frac{1}{1-x}$ for all $x \in(-1,1)$, and then it follows that the derivative of $\frac{1}{1-x}$ is the function defined by $\sum_{n=1}^{\infty} n x^{n-1}$ for all $x \in(-1,1)$. Since we already know that the derivative of $\frac{1}{1-x}$ is $\frac{1}{(1-x)^{2}}$ we have $\frac{1}{(1-x)^{2}}=\sum_{n=1}^{\infty} n x^{n-1}$ for all $x \in(-1,1)$.

EXERCISE 91. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x)=\sum_{n=0}^{\infty}(-1)^{n} \frac{x^{2 n}}{(2 n)!}$. Show that $f$ is the solution of the initial value problem

$$
\left\{\begin{array}{l}
y^{\prime \prime}+y=0 \\
y(0)=1 \\
y^{\prime}(0)=0
\end{array}\right.
$$

Solution:

EXERCISE 92. Let $f(x)=\sum_{n=1}^{\infty} \frac{(2 x+1)^{n}}{n}$. Determine the interval of convergence of $f$, its derivative, and all its antiderivatives.

Solution:

EXERCISE 93. Determine $a, b \in \mathbb{R}$ and $\left(c_{n}\right)_{n=1}^{\infty}$ a sequence of real numbers such that for all $x \in[a, b] \arctan (x)=\sum_{n=0}^{\infty} c_{n} x^{n}$, and deduce that $\frac{\pi}{4}=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2 n+1}$.

## Solution:

## 3. Power series representation of functions

In the previous section we have already realized that representing a function as a power series is a powerful tool. In this section we investigate what functions admit a power series representation, also called a power series expansion. We start the the following simple but basic observation, which states that functions that admit a power series expansion are very regular (differentiable infinitely many times) and the coefficients in the expansion are determined by the values of the derivatives of the functions.

Theorem 33. Let $\sum_{n=0}^{\infty} c_{n}(x-a)^{n}$ be a power series centered at a and with radius of convergence $R$ and consider the function $f:(a-R, a+R) \rightarrow \mathbb{R}$, defined by $f(x)=\sum_{n=0}^{\infty} c_{n}(x-a)^{n}$. Then, $f$ is differentiable infinitely many times on $(a-R, a+R)$ and necessarily $c_{n}=\frac{f^{(n)}(a)}{n!}$.

The previous theorem reduces the study of functions admitting a power series representation to the study of a power series that can be associated to every function that is differentiable infinitely many times, namely its Taylor series.

Definition 24. Let $r>0$ and $f:(a-r, a+r) \rightarrow \mathbb{R}$ be a function that has derivative of all orders at $x=a$.

- The polynomial of degree $n$,

$$
T_{n}(f)(x)=\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^{k}
$$

is called the $n$-th degree Taylor polynomial of $f$ at $a$.

- The power series

$$
T(f)(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^{n}
$$

is called the Taylor series of $f$ at $a$.

- The function $R_{n}:(a-r, a+r) \rightarrow \mathbb{R}$ defined by

$$
R_{n}(x)=f(x)-T_{n}(f)(x)
$$

is the Taylor remainder of order $n$ of $f$.
Remark 16. The Taylor series of $f$ at $a$ is defined as the point-wise limit of the Taylor polynomials of $f$ at $a$, i.e., for all $x$ in the interval of convergence of the Taylor series, $T(f)(x)=\lim _{n \rightarrow \infty} T_{n}(f)(x)$. When $a=0$ the Taylor series of $f$ at 0 is simply called the Maclaurin series of $f$.

Example 26. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=e^{2 x}$. Then, $f$ has derivatives of all orders everywhere $f^{(k)}(x)=2^{k} e^{2 x}$ and thus $T_{n}(f)(x)=\sum_{k=0}^{n} \frac{2^{k} e^{2 a}}{k!}(x-a)^{k}$ is the $n$-th degree Taylor polynomial of $f$ at $a$. Using the ratio test we can show that the Taylor series of $f$ at $a$ exists for all $x \in \mathbb{R}$ and $T(f)(x)=\sum_{n=0}^{\infty} \frac{2^{n} e^{2 a}}{n!}(x-a)^{n}$.

Exercise 94. Let $a>0$ and $f:(0, \infty) \rightarrow \mathbb{R}$ defined by $f(x)=\ln (x)$. Show that the Taylor series of $f$ at $a$ has radius of convergence $R=a$, and that its interval of convergence is $(0,2 a]$.

## Solution:

As we have seen in the introduction of this section, if a function $f$ has a representation as a power series then this representation is unique and is given by the Taylor series of $f$. Therefore a function $f$ has a power series expansion if $f$ is equal to its Taylor series.

Theorem 34 (Remainder criterion). Let $a \in \mathbb{R}$ and $f$ be a real-valued function. If there exists $r>0$ such that
(1) $f$ is differentiable infinitely many times on $(a-r, a+r)$ and
(2) for all $x \in(a-r, a+r) \lim _{n \rightarrow \infty} R_{n}(f)(x)=0$,
then for all $x \in(a-r, a+r)$

$$
f(x)=T(f)(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^{n}=\lim _{n \rightarrow \infty} T_{n}(f)(x)
$$

In practice, to apply the remainder criterion we will use Taylor's inequality.
Theorem 35 (Taylor's Inequality). Let $a \in \mathbb{R}$ and $f$ be a real-valued function. If there exist $r>0$ such that
(1) $f$ is differentiable infinitely many times on $(a-r, a+r)$, and
(2) for all $n \geq 0$ there exits $M \geq 0$ such for all $x \in(a-r, a+r),\left|f^{(n+1)}(x)\right| \leq$ M,
then for all $x \in(a-r, a+r)$ and $n \geq 0$,

$$
\left|R_{n}(f)(x)\right| \leq \frac{M}{(n+1)!}|x-a|^{n+1}
$$

Example 27. Let $a \in \mathbb{R}$. Using Taylor's inequality we can show that for all $x \in \mathbb{R}$,

$$
e^{2 x}=\sum_{n=0}^{\infty} \frac{2^{n} e^{2 a}}{n!}(x-a)^{n}
$$

EXERCISE 95. Show that for all $x \in \mathbb{R}, e^{x}=\sum_{n=0}^{\infty} \frac{x^{n}}{n!}$, and deduce that

$$
e=\sum_{n=0}^{\infty} \frac{1}{n!}
$$

Solution:

ExErcise 96. Show that for all $x \in \mathbb{R}, e^{x}=\sum_{n=0}^{\infty} \frac{x^{n}}{n!}$, and deduce that

$$
e=\sum_{n=0}^{\infty} \frac{1}{n!} .
$$

Solution:

Exercise 97. Show that for all $x \in \mathbb{R}, e^{3 x}=\sum_{n=0}^{\infty} \frac{3^{n} e^{6}}{n!}(x-2)^{n}$.
Solution:

Exercise 98. Show that for all $x \in \mathbb{R}, \sin (x)=\sum_{n=0}^{\infty}(-1)^{n} \frac{x^{2 n+1}}{(2 n+1)!}$.
Solution:

EXERCISE 99. Show that for all $x \in \mathbb{R}, \cos (x)=\sum_{n=0}^{\infty}(-1)^{n} \frac{x^{2 n}}{(2 n)!}$.
Solution:

Exercise 100. Let $a>0$. Show that for all $x \in(0,2 a]$,

$$
\ln (x)=\ln (a)+\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{a^{n} n}(x-a)^{n}
$$

Solution:

