Example – 8 February 2008. At the end of class I did an example that had some substitution errors in it. Here it is correctly done.

Problem. *Find $\partial z/\partial x$ and $\partial z/\partial y$ for the function implicitly defined by the equation*

$$F(x,y,z) = x^5 - 3x^2z + z^2 - xyz^6 + 2 = 0. \tag{1}$$

In addition, find the tangent plane to this function at the point $x = 1$, $y = 1$, $z = 1$, given that $(1,1,1)$ satisfies the equation $F(1,1,1) = 0$.

Solution. Suppose that we can solve $F(x,y,z) = 0$ for $z = z(x,y)$; that is, plugging $z = z(x,y)$ into F results in the equation $F(x,y,z(x,y)) = 0$. Applying the chain rule to this gives us the following:

$$\begin{aligned}\frac{\partial 0}{\partial x} = 0 \quad &= \quad \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\ &= \quad (5x^4 - 6xz - yz^6) + (-3x^2 + 2z - 6xyz^5)\frac{\partial z}{\partial x}.\end{aligned}$$

Next, solve the last equation for the partial $z_x = \frac{\partial z}{\partial x}$ to get

$$\frac{\partial z}{\partial x} = -\frac{5x^4 - 6xz - yz^6}{-3x^2 + 2z - 6xyz^5} = \frac{5x^4 - 6xz - yz^6}{3x^2 - 2z + 6xyz^5}. \tag{2}$$

To get z_y, we do the same steps:

$$\begin{aligned}\frac{\partial 0}{\partial y} = 0 \quad &= \quad \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\ &= \quad (-yz^6) + (-3x^2 + 2z - 6xyz^5)\frac{\partial z}{\partial y}.\end{aligned}$$

Solving for z_y then gives us

$$\frac{\partial z}{\partial y} = -\frac{-xz^6}{-3x^2 + 2z - 6xyz^5} = -\frac{yz^6}{3x^2 - 2z + 6xyz^5}. \tag{3}$$

Notice that when we use $x = 1$, $y = 1$, and $z = 1$, we get

$$z_x(1,1) = \left.\frac{5x^4 - 6xz - yz^6}{3x^2 - 2z + 6xyz^5}\right|_{(1,1,1)} = -\frac{2}{7}.$$

Similarly, we see that

$$z_y(1,1) = -\left.\frac{yz^6}{3x^2 - 2z + 6xyz^5}\right|_{(1,1,1)} = -\frac{1}{7}.$$

The equation of the tangent plane is to $z = f(x,y)$ at (x_0, y_0, z_0) is $z = z_0 + z_x(x_0,y_0)(x - x_0) + z_y(x_0,y_0)(y - y_0)$. Hence, the tangent plane to the function implicitly defined by (1) at $(1,1,1)$ is

$$z = 1 - \frac{2}{7}(x-1) - \frac{1}{7}(y-1) = \frac{10}{7} - \frac{2}{7}x - \frac{1}{7}y. \tag{4}$$