Example – 8 February 2008. At the end of class I did an example that had some substitution errors in it. Here it is correctly done.

Problem. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the function implicitly defined by the equation

$$F(x, y, z) = x^5 - 3x^2z + z^2 - xyz^6 + 2 = 0. \quad (1)$$

In addition, find the tangent plane to this function at the point $x = 1, y = 1, z = 1$, given that $(1, 1, 1)$ satisfies the equation $F(1, 1, 1) = 0$.

Solution. Suppose that we can solve $F(x, y, z) = 0$ for $z = z(x, y)$; that is, plugging $z = z(x, y)$ into $F$ results in the equation $F(x, y, z(x, y)) = 0$. Applying the chain rule to this gives us the following:

$$\frac{\partial}{\partial x} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = (5x^4 - 6xz - yz^6) + (-3x^2 + 2z - 6xyz^5) \frac{\partial z}{\partial x}. \quad (2)$$

Next, solve the last equation for the partial $z_x = \frac{\partial z}{\partial x}$ to get

$$\frac{\partial z}{\partial x} = \frac{-5x^4 - 6xz - yz^6}{-3x^2 + 2z - 6xyz^5} = \frac{5x^4 - 6xz - yz^6}{3x^2 - 2z + 6xyz^5}. \quad (3)$$

To get $z_y$, we do the same steps:

$$\frac{\partial}{\partial y} = \frac{\partial F}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = (-yz^6) + (-3x^2 + 2z - 6xyz^5) \frac{\partial z}{\partial y}. \quad (4)$$

Solving for $z_y$ then gives us

$$\frac{\partial z}{\partial y} = -\frac{-3x^2 + 2z - 6xyz^5}{-3x^2 + 2z - 6xyz^5} = -\frac{yz^6}{3x^2 - 2z + 6xyz^5}. \quad (5)$$

Notice that when we use $x = 1, y = 1$, and $z = 1$, we get

$$z_x(1, 1) = \frac{5x^4 - 6xz - yz^6}{3x^2 - 2z + 6xyz^5} \bigg|_{(1,1,1)} = \frac{2}{7}. \quad (6)$$

Similarly, we see that

$$z_y(1, 1) = -\frac{yz^6}{3x^2 - 2z + 6xyz^5} \bigg|_{(1,1,1)} = -\frac{1}{7}. \quad (7)$$

The equation of the tangent plane is to $z = f(x, y)$ at $(x_0, y_0, z_0)$ is $z = z_0 + z_x(x_0, y_0)(x - x_0) + z_y(x_0, y_0)(y - y_0)$. Hence, the tangent plane to the function implicitly defined by (1) at $(1, 1, 1)$ is

$$z = 1 - \frac{2}{7}(x - 1) - \frac{1}{7}(y - 1) = \frac{10}{7} - \frac{2}{7}x - \frac{1}{7}y. \quad (8)$$