Exercise 1, Chapter 4. (Math 414-501, Spring 2010)

The function \( f(x) \) is given by

\[
\begin{align*}
  f(x) &= \begin{cases} 
    -1, & 0 \leq x < 1/4, \\
    4, & 1/4 \leq x < 1/2, \\
    2, & 1/2 \leq x < 3/4, \\
    -3, & 3/4 \leq x < 1, \\
    0, & \text{otherwise}. 
  \end{cases}
\end{align*}
\]

Since \( f \) is in \( V_2 \), we can write in terms of the basis \( \{ \phi(2^k x - k) \}_{k=0}^3 \) (cf. Definition 4.1 in the text):

\[
  f(x) = -\phi(4x) + 4\phi(4x - 1) + 2\phi(4x - 2) - 3\phi(4x - 3).
\]

The easiest way to approach decomposing \( f \) into its components along \( V_0, W_0, \) and \( W_1 \) is to use Lemma 4.10, which states that

\[
\begin{align*}
  \phi(2^j x) &= (\psi(2^{j-1} x) + \phi(2^{j-1} x))/2, \\
  \phi(2^j x - 1) &= (\phi(2^{j-1} x) - \psi(2^{j-1} x))/2.
\end{align*}
\]

Begin by getting the \( V_1, W_1 \) parts. To do this, replace the functions \( \phi(4x - k) \) as follows:

\[
\begin{align*}
  \phi(4x) &= (\phi(2x) + \psi(2x))/2, \\
  \phi(4x - 1) &= (\phi(2x) - \psi(2x))/2, \\
  \phi(4x - 2) &= (\phi(2x - 1) + \psi(2x - 1))/2, \\
  \phi(4x - 3) &= (\phi(2x - 1) - \psi(2x - 1))/2.
\end{align*}
\]

Using these, put \( f \) into the form

\[
  f(x) = (\frac{1}{2} + 2)\phi(2x) + (1 - 3/2)\phi(2x - 1) + (\frac{1}{2} - 2)\psi(2x) + (1 + 3/2)\psi(2x - 1)
\]

\[
= \frac{3}{2} \phi(2x) - \frac{1}{2} \phi(2x - 1) - \frac{5}{2} \psi(2x) + \frac{5}{2} \psi(2x - 1).
\]

Finally, use \( \phi(2x) = (\phi(x) + \psi(x))/2 \) and \( \phi(2x - 1) = (\phi(x) - \psi(x))/2 \) in the equation above to obtain

\[
  f(x) = \frac{1}{2} \phi(x) + \psi(x) - \frac{5}{2} \psi(2x) + \frac{5}{2} \psi(2x - 1),
\]

which is what was asked for.