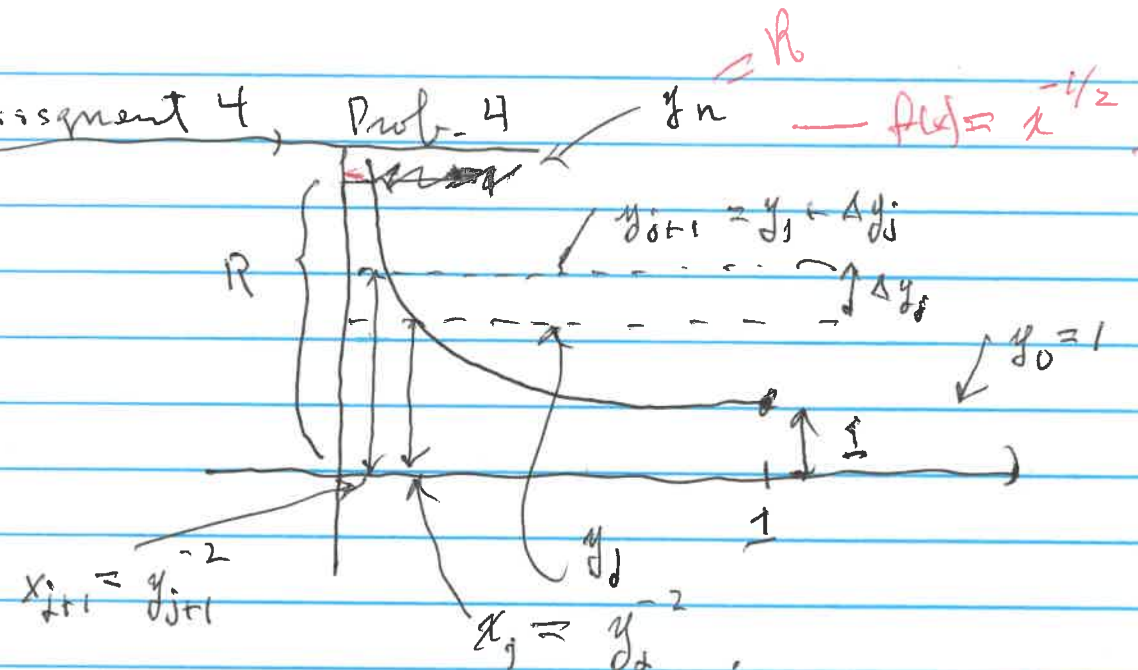
Assignment 4, Prob. 4 — $f(x) = x^{-1/2}$



Simple function: // Lebesgue.

$$\int s = \sum_{j=0}^{n-1} y_j^* \, \underbrace{\mu\left(f^{-1}\left([y_j, y_{j+1})\right)\right)}_{x_j - x_{j+1}},$$

Choose $y_j^* = y_j$. Then $s_* \le f \quad \forall\, x \in [0,1]$.

$$\Rightarrow \int s_* = \sum_{j=0}^{n-1} y_j \left(y_j^{-2} - y_{j+1}^{-2}\right) = \sum_{j=0}^{n-1} \left(y_j^{-1} - y_{j+1}^{-1}\right)(y_j)\left(y_j^{-1} + y_{j+1}^{-1}\right)$$

$$\int s_* = \sum_{j=0}^{n-1} \left(y_j^{-1} - y_{j+1}^{-1}\right)\left(1 + \frac{y_{j+1} - \Delta y_j}{y_{j+1}}\right)$$

$$\Rightarrow \int s_* = \sum_{j=0}^{n-1} \underbrace{\left(y_j^{-1} - y_{j+1}^{-1}\right)}_{+} \underbrace{\left(2 - \Delta y_j \cdot y_{j+1}^{-1}\right)}_{+}$$

1) Upper bnd. $\displaystyle \int s_* \le 2 \sum_{j=0}^{n-1} \left(y_j^{-1} - y_{j+1}^{-1}\right) \le 2\left(1 - \frac{1}{R}\right) \le 2$

2) Lower bnd $\quad y_{j+1} > y_j > \cdots y_1 > y_0 = 1$. Then,

$$\frac{\Delta y_j}{y_{j+1}} \le \Delta y_j \quad \& \quad 2 - \frac{\Delta y_j}{y_{j+1}} < 2 - \Delta y_j \}$$

$$\Rightarrow 2 - \frac{\Delta y_j}{y_{j+1}} > 2 - \|P\|. \quad \Rightarrow \quad s_* \ge \left(2 - \|P\|\right) \sum_{j=0}^{n-1} \left(y_j^{-1} - y_{j+1}^{-1}\right)$$

$$\cancel{\Rightarrow \quad S_x \le 2\left(\sum_{j=0}^{n-1}\right.}$$

$$(2-\|P\|)\sum_{j=0}^{n-1}(\text{SAME}) \le S_x \le 2\left(\sum_{j=0}^{n-1}\left(y_j^{-1}-y_{j+1}^{-1}\right)\right)$$

Evaluate $\sum_{j=0}^{n-1} y_j^{-1} - y_{j+1}$.

$$\sum_{j=0}^{n-1} y_j^{-1}-y_{j+1}^{-1} = y_0^{-1}+\cancel{y_1^{-1}}+\cdots+\cancel{y_{n-1}^{-1}} - \cancel{y_1^{-1}}-\cancel{y_2^{-1}}-\cdots-y_n^{-1} = y_0^{-1}-y_n^{-1}$$

$$" \quad = 1-\frac{1}{R}.$$

$$\therefore \quad (2-\|P\|)\left(1-\frac{1}{R}\right) \le S_x \le 2\left(1-\frac{1}{R}\right) \le 2$$

For the partition P, the Δy_j's are essentially arbitrary. They only depend (roughly) on the # of pts in P. Thus, for every ~~P, we may~~ $\delta>0$ we may choose P so that $\|P\|<\delta$.

$$\Rightarrow \quad (2-\delta)\left(1-\frac{1}{R}\right) < (2-\|P\|)\left(1-\frac{1}{R}\right) \le S_x \le 2\left(1-\frac{1}{R}\right), \quad \forall R \ge 1.$$

$$\Rightarrow \quad (2-\delta)\left(1-\frac{1}{R}\right) < S_x < 2\left(1-\frac{1}{R}\right) \le 2.$$

indep. of P, ~~and~~ Holds $\forall P$ w/ $\|P\|<\delta$.

$$\Rightarrow \quad (2-\delta)\sup\left(1-\frac{1}{R}\right) < \sup S_x \le 2$$

$$\Rightarrow \quad (2-\delta) \le \sup S_x \le 2, \quad \forall \delta$$

$$\Rightarrow \quad 2 \le \sup S_x \le 2 \Rightarrow \int_0^1 x^{-k}\,dx = 2.$$