The Closed Range Theorem
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Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ denote the bounded linear operators on $\mathcal{H}$ and the compact operators on $\mathcal{H}$, respectively.

Our aim here is to prove that the range of an operator of the form
\[ L = I - \lambda K, \]
where $K$ is compact, is closed. We will begin with this lemma.

**Lemma 0.1.** Let $K \in \mathcal{C}(\mathcal{H})$, $\lambda \in \mathbb{C}$, and $L = I - \lambda K$. If $f \in R(L^*)$, then there is a constant $c > 0$, independent of $f$, such that
\[ \|Lf\| \geq c\|f\|. \]

**Proof.** If not, then there exists a sequence $\{f_n\}_{n=1}^\infty \subset R(L^*)$ such that $\|f_n\| = 1$ and $\|Lf_n\| \to 0$ as $n \to \infty$. Note that $Lf_n = f_n - \lambda Kf_n$, so $f_n = \lambda Kf_n + Lf_n$. Since the $f_n$’s are bounded and $K$ is compact, we may choose a subsequence $\{f_{n_k}\}$ such that $\{Kf_{n_k}\}$ is convergent. Thus,
\[ \lim_{k \to \infty} f_{n_k} = \lambda \lim_{k \to \infty} Kf_{n_k} + \lim_{k \to \infty} Lf_{n_k}. \]
Both terms on the right are convergent, so $f_{n_k}$ is also convergent. Let $f = \lim_{k \to \infty} f_{n_k}$. By the previous equation, we have that $f = \lambda Kf$, so $Lf = 0$ – i.e., $f \in N(L)$. In addition, because $R(L^*)$ is closed, $f \in R(L^*)$. Since these spaces are orthogonal, $f$ is orthogonal to itself and, consequently, $f = 0$. However, $\lim_{n \to \infty} \|f_n\| = 1 = \|f\|$. This is a contradiction. \[\square\]

**Theorem 0.2** (Closed Range Theorem). If $K \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, then the range of the operator $L = I - \lambda K$ is closed.

**Proof.** We want to show that if there is sequence $\{f_n\} \subset R(L)$ such that $g_n \to g$, then $g = Lf$ for some $f \in \mathcal{H}$. To begin, note that the solution $f_n$ to $g_n = Lf_n$ is not unique if $N(L) \neq \{0\}$. Since $\mathcal{H} = N(L) \oplus \overline{R(L^*)}$, with the two spaces being orthogonal, we may make a unique choice by requiring that $f_n$ be in $\overline{R(L^*)}$. Lemma 0.1 then implies that $\|g_n - g_m\| = L(f_n - f_m) \geq c\|f_n - f_m\|$. Because the convergent sequence $\{g_n\}$ is Cauchy, this inequality also implies that $\{f_n\}$ is Cauchy. Thus, $\{f_n\}$ is convergent to some $f \in \mathcal{H}$. It follows that $g = \lim_{n \to \infty} Lf_n = Lf$, so $g \in R(L)$. \[\square\]

The Closed Range Theorem allows us to apply the Fredholm alternative to equations of the form $u - \lambda Ku = f$. Thus, we have the following result, which applies, for example, to finite rank or Hilbert-Schmidt operators.

**Corollary 0.3.** Let $K \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. The equation $u - \lambda Ku = f$ has a solution if and only if $f \in N(I - \lambda K^*)^\perp$. 1