Throughout these notes, $\mathcal{H}$ denotes a separable Hilbert space. We will use the notation $\mathcal{B}(\mathcal{H})$ to denote the set of bounded linear operators on $\mathcal{H}$. We also note that $\mathcal{B}(\mathcal{H})$ is a Banach space, under the usual operator norm.

1 Compact and Precompact Subsets of $\mathcal{H}$

**Definition 1.1.** A subset $S$ of $\mathcal{H}$ is said to be compact if and only if it is closed and every sequence in $S$ has a convergent subsequence. $S$ is said to be precompact if its closure is compact.

**Proposition 1.2.** Here are some important properties of compact sets.

1. Every compact set is bounded.

2. Let $S$ be bounded set. Then $S$ is precompact if and only if every sequence has a convergent subsequence.

3. Let $\mathcal{H}$ be finite dimensional. Every closed, bounded subset of $\mathcal{H}$ is compact.

4. In an infinite dimensional space, closed and bounded is not enough.

*Proof.* Properties 2 and 3 are left to the reader. For property 1, assume that $S$ is an unbounded compact set. Since $S$ is unbounded, we may select a sequence $\{v_n\}_{n=1}^{\infty}$ from $S$ such that $\|v_n\| \to \infty$ as $n \to \infty$. Since $S$ is compact, this sequence will have a convergent subsequence, say $\{v_{n_k}\}_{k=1}^{\infty}$, which still be unbounded. (Why?) Let $v = \lim_{k \to \infty} v_{n_k}$. Thus, for $\varepsilon = 1$ there is a positive integer $K$ for which $\|v - v_{n_k}\| < 1$ for all $k \geq K$. By the triangle inequality, $\|v_{n_k}\| \leq \|v\| + 1$. Now, the right side is bounded, but the left side isn’t, since $\|v_{n_k}\| \to \infty$ as $k \to \infty$. This is a contradiction, so $S$ must be bounded. For property 4, let $S = \{f \in \mathcal{H} : \|f\| \leq 1\}$. Every o.n. basis $\{\phi_n\}_{n=1}^{\infty}$ is in $S$. However, for such a basis $\|\phi_m - \phi_n\| = \sqrt{2}$, $n \neq m$. Again, this means there are no Cauchy subsequences in $\{\phi_n\}_{n=1}^{\infty}$, and consequently, no convergent subsequences. Thus, $S$ is not compact. \(\square\)
Compact Operators

Definition 2.1. Let $K : \mathcal{H} \to \mathcal{H}$ be linear. $K$ is said to be compact if and only if $K$ maps bounded sets into precompact sets. Equivalently, $K$ is compact if and only if for every bounded sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ the sequence $\{Kv_n\}_{n=1}^{\infty}$ has a convergent subsequence. We denote the set of compact operators on $\mathcal{H}$ by $C(\mathcal{H})$.

Proposition 2.2. If $K \in C(\mathcal{H})$, then $K$ is bounded – i.e., $C(\mathcal{H}) \subset B(\mathcal{H})$. In addition, $C(\mathcal{H})$ is a subspace of $B(\mathcal{H})$.

Proof. We leave this as an exercise for the reader.

We now turn to giving some examples of compact operators. We start with the finite-rank operators. If the range of a bounded operator $K$ is finite dimensional, then we say that $K$ is a finite-rank operator.

Proposition 2.3. Every finite-rank operator $K$ is compact.

Proof. The range of $K$ is finite dimensional, so every bounded subset of the range is precompact. Let $S \subseteq \{f \in \mathcal{H} : \|f\| \leq C\}$, where $C$ is fixed. Note that the range of $K$ restricted to $S$ is also bounded: $\|Kf\| \leq \|K\|_{op}\|f\| \leq C\|K\|_{op}$. Thus, $K$ maps a bounded set $S$ into a bounded subset of a finite dimensional subspace of $\mathcal{H}$, which is itself precompact. Hence, $K$ is thus compact.

To describe $K$ explicitly, let $\{\phi_k\}_{k=1}^{n}$ be a basis for $R(K)$. Then, $Kf = \sum_{k=1}^{n} a_k \phi_k$. We want to see how the $a_k$’s depend on $f$. Consider $\langle Kf, \phi_j \rangle = \langle f, K^* \phi_j \rangle = \sum_{k=1}^{n} a_k \langle \phi_k, \phi_j \rangle$. Next let $\psi_j = K^* \phi_j$, so that $\langle f, K^* \phi_j \rangle = \langle f, \psi_j \rangle$. Because $\{\phi_k\}_{k=1}^{n}$ is a basis, it is linear independent. Hence, the Gram matrix $G_{j,k} = \langle \phi_k, \phi_j \rangle$ is invertible, and so we can solve the system of equations $\langle f, \psi_j \rangle = \sum_{k=1}^{n} G_{j,k} a_k$. Doing so yields $a_k = \sum_{j=1}^{n} (G^{-1})_{k,j} \langle f, \psi_j \rangle$. The $a_k$’s are obviously linear in $f$. Of course, a different basis will give a different representation.

Let $\mathcal{H} = L^2[0,1]$. A particularly important set of finite rank operators in $C(\mathcal{H})$ are ones given by finite rank or degenerate kernels, $k(x, y) = \sum_{k=1}^{n} \phi_k(x) \overline{\psi}_k(y)$, where the functions involved are in $L^2$. The operator is then $Kf(x) = \int_{0}^{1} k(x, y)f(y)dy$. In the example that we did for resolvents, the kernel was $k(x, y) = xy^2$, and the operator was $Ku(x) = \int_{0}^{1} k(x, y)u(y)dy$. Later, we will show that the Hilbert-Schmidt kernels also yield compact operators. Before, we do so, we will discuss a few more properties of compact operators.
Lemma 2.4. Let \( \{ \phi_n \}_{n=1}^{\infty} \) be an o.n. set in \( \mathcal{H} \) and let \( K \in \mathcal{C}(\mathcal{H}) \). Then, \( \lim_{n \to \infty} K\phi_n = 0 \).

Proof. Suppose not. Then we may select a subsequence \( \{ \phi_m \} \) of \( \{ \phi_n \}_{n=1}^{\infty} \) for which \( \| K\phi_m \| \geq \alpha > 0 \) for all \( m \). Because \( K \) is compact, we can also select a subsequence \( \{ \phi_k \} \) of \( \{ \phi_m \} \) for which \( K\phi_k \) is convergent to \( \psi \in \mathcal{H} \). Now, \( \{ \phi_k \} \) being a subsequence of \( \{ \phi_m \} \) implies that \( \| K\phi_k \| \geq \alpha > 0 \). Taking the limit in this inequality yields \( \| \psi \| \geq \alpha > 0 \). Next, note that \( \lim_{k \to \infty} \langle K\phi_k, \psi \rangle = \| \psi \|^2 \). However, \( \lim_{k \to \infty} \langle K\phi_k, \psi \rangle = \lim_{k \to \infty} \langle \phi_k, K^* \psi \rangle = 0 \), by Bessel’s inequality. Thus, \( \| \psi \|^2 = 0 \), which is a contradiction. \( \square \)

This lemma is a special case of a more general result. We say that a sequence \( \{ f_n \} \) is weakly convergent to a \( f \in \mathcal{H} \) if and only if for all \( g \in \mathcal{H} \) we have \( \lim_{n \to \infty} \langle f_n, g \rangle = \langle f, g \rangle \). For example, the o.n. set in the lemma weakly converges to 0.

Proposition 2.5. Let \( \{ f_n \} \) weakly converge to \( f \in \mathcal{H} \). If \( K \in \mathcal{C}(\mathcal{H}) \), then \( \lim_{n \to \infty} Kf_n = Kf \). That is, \( K \) maps weakly convergent sequences into “strongly” convergent ones.

Proof. The proof is similar to that of Lemma 2.4. See exercise 4 in assignment 9. \( \square \)

We remark that the converse is true, too. This leads to an alternative characterization of compact operators: \( K \) is compact if and only if \( K \) maps weakly convergent sequences into strongly convergent ones. See the book *Functional Analysis*, by F. Riesz and B. Sz.-Nagy.

Our next result is one of the most important theorems in the theory of compact operators.

Theorem 2.6. \( \mathcal{C}(\mathcal{H}) \) is a closed subspace of \( \mathcal{B}(\mathcal{H}) \).

Proof. Suppose that \( \{ K_n \}_{n=1}^{\infty} \) is a sequence in \( \mathcal{C}(\mathcal{H}) \) that converges to \( K \in \mathcal{B}(\mathcal{H}) \), in the operator norm. We want to show that \( K \) is compact. Assume the \( \{ v_k \} \) is a bounded sequence in \( \mathcal{H} \), with \( \| v_k \| \leq C \) for all \( k \). Compactness will follow if we can prove that \( \{ Kv_k \} \) has a convergent subsequence. The technique for doing this is often called a diagonalization argument. We start with the full sequence and form \( \{ K_1 v_k \} \). Since \( K_1 \) is compact, we can select a subsequence \( \{ v_k^{(1)} \} \) such that \( \{ K_1 v_k^{(1)} \} \) is convergent. We may carry out the same procedure with \( \{ K_2 v_k^{(1)} \} \), selecting a subsequence of \( \{ K_2 v_k^{(1)} \} \) that is convergent. Call it \( \{ v_k^{(2)} \} \). Since this is a subsequence of \( \{ v_k^{(1)} \} \), \( \{ K_1 v_k^{(2)} \} \) is convergent. Continuing in this way, we construct subsequences
\{u_k^{(j)}\} for which \(\{K_m u_k^{(j)}\}\) is convergent for all \(1 \leq m \leq j\). Next, we let \(u_j := u_j^{(j)}\), the “diagonal” sequence. This is a subsequence of all of the \(\{u_k^{(j)}\}'s. Consequently, for \(n\) fixed, \(\{K_n u_j\}_{j=1}^{\infty}\) will be convergent. To finish up, we will use an “up, over, and around” argument. Note that for all \(\|K\|\) that is Cauchy and therefore convergent.

Since \(\|K u_\ell - K u_m\| \leq \|K - K\|_{op}\|u_\ell\| \leq 2C\|K - K\|_{op}\) and, similarly, \(\|K u_m - K u_m\| \leq 2C\|K - K\|_{op}\), so \(\|K u_\ell - K u_m\| \leq 4C\|K - K\|_{op} + \|K u_m - K u_m\|\). Let \(\varepsilon > 0\). First choose \(N\) such that for \(n \geq N\), \(\|K - K\|_{op} < \varepsilon/(8C)\). Fix \(n\). Because \(\{K_n u_\ell\}\) is convergent, it is Cauchy. Choose \(N'\) so large that \(\|K_n u_\ell - K_n u_m\| \leq \varepsilon/2\) for all \(\ell, m \geq N'\). Putting these two together yields \(\|K u_\ell - K u_\ell\| \leq \varepsilon\), provided \(\ell, m \geq N'\). Thus \(\{K u_\ell\}\) is Cauchy and therefore convergent.

**Corollary 2.7.** Hilbert-Schmidt operators are compact.

**Proof.** Let \(H = L^2[0,1]\) and suppose \(k(x, y) \in L^2(R), R = [0,1] \times [0,1]\). The associated Hilbert-Schmidt operator is \(Ku = \int_0^1 k(x, y)u(y)dy\). Let \(\{\phi_n\}_{n=1}^{\infty}\) be an o.n. basis for \(L^2[0,1]\). With a little work, one can show that \(\{\phi_n(x)\phi_m(y)\}_{n,m=1}^{\infty}\) is an o.n. basis for \(L^2(R)\). Also, from example 2 in the notes on Bounded Operators & Closed Subspaces, we have that \(\|K\|_{op} \leq \|k\|_{L^2(R)}\). Expand \(k(x, y)\) in the o.n. basis \(\{\phi_n(x)\phi_m(y)\}_{n,m=1}^{\infty}\):

\[
k(x, y) = \sum_{n,m=1}^{\infty} \alpha_{m,n}\phi_n(x)\phi_m(y), \quad \alpha_{m,n} = \langle k(x, y), \phi_n(x)\phi_m(y) \rangle_{L^2(R)}
\]

Next, let \(k_N(x, y) = \sum_{n,m=1}^{N} \alpha_{m,n}\phi_n(x)\phi_m(y)\) and also \(K_N\) be the finite rank operator \(K_N u(x) = \int_0^1 k_N(x, y)u(y)dy\). By Parseval’s theorem, we have that \(\|k - k_N\|_{L^2(R)}^2 = \sum_{n,m=N+1}^{\infty} |\alpha_{m,n}|^2\) and by example 2 mentioned above, \(\|K - K_N\|_{op}^2 \leq \|k - k_N\|_{L^2(R)}^2\), so

\[
\|K - K_N\|_{op}^2 \leq \sum_{n,m=N+1}^{\infty} |\alpha_{m,n}|^2
\]

Because the series on the right above converges to 0 as \(N \to \infty\), we have \(\lim_{N \to \infty} \|K - K_N\| = 0\). Thus \(K\) is the limit in \(B(L^2[0,1])\) of finite rank operators, which are compact. By the theorem above, \(K\) is also compact. \(\square\)

\(^1\)See Keener, Theorem 3.5
We now turn to some of the algebraic properties of $C(H)$.

**Proposition 2.8.** Let $K \in C(H)$ and let $L \in B(H)$. Then both $KL$ and $LK$ are in $C(H)$.

**Proof.** Let $\{v_k\}$ be a bounded sequence in $H$. Since $L$ is bounded, the sequence $\{Lv_k\}$ is also bounded. Because $K$ is compact, we may find a subsequence of $\{KLv_k\}$ that is convergent, so $KL \in C(H)$. Next, again assuming $\{v_k\}$ is a bounded sequence in $H$, we may extract a convergent subsequence from $\{Kv_k\}$, which, with a slight abuse of notation, we will denote by $\{Kv_j\}$. Because $L$ is bounded, it is also continuous. Thus $\{LKv_j\}$ is convergent. It follows that $LK$ is compact. □

**Proposition 2.9.** $K$ is compact if and only if $K^*$ is compact.

**Proof.** Because $K$ is compact, it is bounded and so is its adjoint $K^*$, in fact $\|K^*\|_\text{op} = \|K\|_\text{op}$. By Proposition 2.8, we thus have that $KK^*$ is compact. It follows that if $\{u_n\}$ be a bounded sequence in $H$, then we may extract a subsequence $\{u_j\}$ such that the sequence $\{KK^*v_j\}$ is convergent. This of course means that this sequence is also Cauchy. Note that

$$\langle KK^*(v_j - v_k), v_j - v_k \rangle = \langle K^*(v_j - v_k), K^*(v_j - v_k) \rangle = \|K^*(v_j - v_k)\|^2. $$

From and the fact that $\{v_j\}$ is bounded, we see that $\langle KK^*(v_j - v_k), v_j - v_k \rangle \leq \|v_j - v_k\| \|KK^*(v_j - v_k)\| \leq C\|KK^*(v_j - v_k)\|$. Thus,

$$\|K^*(v_j - v_k)\|^2 \leq C\|KK^*(v_j - v_k)\|$$

Since $\{KK^*v_j\}$ is Cauchy, for every $\varepsilon > 0$, we can find $N$ such that whenever $j, k \geq N$, $\|KK^*(v_j - v_k)\| < \varepsilon^2/C$. It follows that $\|K^*(v_j - v_k)\| < \varepsilon$, if $j, k \geq N$. This implies that $\{K^*v_j\}$ is Cauchy and therefore convergent. □

We want to put this in more algebraic language. Taking $L$ to be compact in Proposition 2.8, we have that the product of two compact operators is compact. Since $C(H)$ is already a subspace, this implies that it is an algebra. Moreover, by taking $L$ to be just a bounded operator, we have that $C(H)$ is a two-sided *ideal* in the algebra $B(H)$. Since $K$ being compact implies $K^*$ is compact, $C(H)$ is closed under the operation of taking adjoints; thus, $C(H)$ is a *-ideal. Finally, by Theorem 2.6, we have that $C(H)$ is a closed subspace of $B(H)$. We summarize these results as follows.

**Theorem 2.10.** $C(H)$ is a closed, two-sided, *-ideal in $B(H)$.  

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We remark that a closed $\ast$-algebra in $B(\mathcal{H})$ is called a $C^\ast$-algebra. So, $C(\mathcal{H})$ is a $C^\ast$-algebra that is also a two-sided ideal in $B(\mathcal{H})$.