A Resolvent Example
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November, 2014

Problem. Let \( k(x, y) = xy^2 \), \( Ku(x) = \int_0^1 k(x, y)u(y)dy \), and \( Lu = u - \lambda Ku \). Assume that \( L \) has closed range.

1. Determine the values of \( \lambda \) for which \( Lu = f \) has a solution for all \( f \).
   Solve \( Lu = f \) for these values of \( \lambda \).

2. For the remaining values of \( \lambda \), find a condition on \( f \) the guarantees a solution to \( Lu = f \). When \( f \) satisfies this condition, solve \( Lu = f \).

Solution. (1) Because \( R(L) \) is closed, the Fredholm alternative applies. We begin by finding \( N(L^*) \). First, we have that \( L^* = I - \lambda K^* \), where \( K^* w = \int_0^1 k(y, x)w(y)dy = \int_0^1 yx^2w(y)dy \). We want to find all \( w \) for which \( L^* w = w - \bar{\lambda} \int_0^1 x^2yw(y)dy = 0 \). Note that \( w = \bar{\lambda}x^2 \int_0^1 yw(y)dy \), so \( w = Cx^2 \).

Putting this back into the equation for \( w \) yields \( Cx^2 = \bar{\lambda}Cx^2 \int_0^1 y^2dy = C(\bar{\lambda}/4)x^2 \). Thus, \( C = (\bar{\lambda}/4)C \). If \( \bar{\lambda}/4 \neq 1 \), then \( C = 0 \) and \( N(L^*) = \{0\} \). Thus, if \( \bar{\lambda}/4 \neq 1 \) - i.e., \( \lambda \neq 4 \), \( Lu = f \) has a solution for all \( f \in L^2[0,1] \).

To find \( u \), note that \( u - \lambda x \int_0^1 y^2u(y)dy = f \), and so we only need to find \( \int_0^1 y^2u(y)dy \). The trick for doing this is to multiply \( Lu = f \) by \( x^2 \) and then integrate. Doing this results in \( \int_0^1 y^2u(y)dy - \frac{\lambda}{4} \int_0^1 y^2u(y)dy = \int_0^1 y^2f(y)dy \).

From this we get \( \int_0^1 y^2u(y)dy = \frac{1}{1-\lambda/4} \int_0^1 y^2f(y)dy \). Finally, we arrive at

\[
    u(x) = f(x) + \frac{4\lambda}{4-\lambda} x \int_0^1 y^2f(y)dy = f(x) + \frac{4\lambda}{4-\lambda} Kf(x).
\]

In operator form,

\[
(I - \lambda K)^{-1} = I + \frac{4\lambda}{4-\lambda} K.
\]

The operator \((I - \lambda K)^{-1}\) is called the resolvent of \( K \).

(2) When \( \lambda = 4 \), \( N(L^*) = \text{span}\{x^2\} \). By the Fredholm alternative, \( Lu = f \) has a solution if and only if \( \int_0^1 x^2f(x)dx = 0 \). To solve \( u - 4x \int_0^1 y^2u(y)dy = f \) for \( u \), we first note that \( \int_0^1 y^2u(y)dy \) is not determined, because \( \int_0^1 y^2u(y)dy - \frac{4}{4} \int_0^1 y^2u(y)dy = \int_0^1 y^2f(y)dy = 0 \). This really just says that have consistency. The constant \( C = \int_0^1 y^2u(y)dy \) is thus arbitrary, and \( u(x) = f(x) + Cx \).