Math 642 Apr. 11, 2016

Take-home-problem 4(a). (Test 1)

Given: f is entire and $|f(z)| \le Ae^{\sigma|z|}$, for all $z \in \mathbb{C}$. Also, on the real axis, $z = x$, $f(x) \in L^\infty$, so $|f(x)| \le B$, $x \in \mathbb{R}$.

Show: $|f(z)| \le Ae^{\sigma|y|}$.

(a) Let ~~$f(z)$~~ $F(z) = f(z)e^{i\sigma z}$. Then,

for $z = x$, $|F(x)| = |f(x)e^{i\sigma x}| = |f(x)| \cdot 1 \le B$.

For $z = iy$, $|F(iy)| = |f(iy) \cdot e^{-\sigma y}| = |f(iy)|e^{-\sigma y}$

$$\Rightarrow \quad |F(iy)| \le Ae^{\sigma|iy|} \cdot e^{-\sigma y} = Ae^{\sigma y - \sigma y} = A.$$

Then, $|F(z)|$ is bounded on $z = iy$, $y \ge 0$ ~~and~~ and $z = x$, $x \ge 0$. ~~In addition~~ Also, for all z in the first quadrant,

$$|F(z)| = |f(z)|e^{-\sigma y} \le Ae^{\sigma r - \sigma y} \le Ae^{\sigma r}.$$

(b) Rotate coords. ~~so that~~ by $\pi/4$. Define

$$\tilde{F}(z) = F(e^{+i\pi/4}z).$$

From (a), on $z = re^{i\pi/4}$ & $z = re^{-\pi/4\, i}$, $\tilde{F}(z)$ is bounded and satisfies

$$|\tilde{F}(z)| \le Ce^{\sigma|z|}$$

on the sector $-\pi/4 \le \theta \le \pi/4$.

(c) Let $G(z) = \tilde{F}(z) e^{-\varepsilon z^{3/2}\sigma}$, $z = re^{i\theta}$, ~~$-\frac{\pi}{4} \le \theta \le \pi/4$~~
$-\frac{\pi}{4} \le \theta \le \pi/4$.

$$\Rightarrow \quad |G(z)| = |\tilde{F}(z)| \cdot \exp\left(-\varepsilon\sigma \mathrm{Re}\left(z^{3/2}\right)\right)$$

$$|G(z)| = |\tilde{F}(z)| \exp\left(-\varepsilon r^{3/2} \cos\left(\frac{3\theta}{2}\right)\sigma\right)$$

$$\Rightarrow \cancel{|G(z)| \le C}$$

$$|G(z)| \le C e^{\sigma r - \varepsilon r^{3/2} \cos\left(\frac{3\theta}{2}\right)\sigma}$$

$$(*) \quad |G(z)| \le C \exp\left(\sigma r^{3/2}\left(r^{-1/2} - \varepsilon \cos\left(\frac{3\theta}{2}\right)\right)\right)$$

Since $\cos\left(\frac{3\theta}{2}\right) \ge \cos\left(\frac{3\pi}{8}\right)$ in $|\theta| \le \pi/4$, ~~we have~~ we have

$$r^{-1/2} - \varepsilon \cos\left(\frac{3\theta}{2}\right) \le r^{-1/2} - \varepsilon \cos\left(\frac{3\pi}{8}\right).$$

Finally, consider the ~~...~~ sector ~~with~~ bounded by the rays along $e^{+\frac{\pi i}{4}}$ & $e^{-\frac{i\pi}{4}}$ ~~and~~ and $r = R$, where R is chosen so large that

$$R^{-1/2} - \varepsilon \cos\left(\frac{3\pi}{8}\right) < 0.$$

From $(*)$, on $|z| = R$, $|G(z)| \le C$. On the two rays $re^{i\pi/4}$, $re^{-i\pi/4}$, $r \le R$, $G(z)$ is also bounded. By the max. modulus theorem,

$$|G(z)| \le D \quad \text{in the sector } |\theta| \le \pi/4.$$

(d) $\tilde{F}(z) = G(z)\, e^{\varepsilon z^{3/2}\sigma}$

$$\Rightarrow |\tilde{F}(z)| = \underbrace{|G(z)|}_{\text{Bounded}}\, e^{\varepsilon r^{3/2}\sin(\frac{3\theta}{2})\sigma}$$

$$\underbrace{|\tilde{F}(z)|}_{\substack{\text{Independent} \\ \text{of } \varepsilon}} \leq \underbrace{D e^{\varepsilon r^{3/2}\sin(\frac{3\theta}{2})\sigma}}_{\downarrow\ \varepsilon \downarrow 0^+ \\ D},$$

$\therefore$

$$|\tilde{F}(z)| \leq D \text{ on } |\theta| \leq \pi/4.$$

Now, rotate by $+\pi/4$,

$$F(z) = \tilde{F}(e^{-\frac{i\pi}{4}} z),$$

Obviously, we have $|F(z)| \leq D$ on $0 \leq \theta \leq \pi/2$

– i.e., 1st quadrant.

(e) $$f(z) = e^{-i\sigma z} F(z)$$

$$\Rightarrow |f(z)| = |F(z)|\, e^{\sigma y} \leq D e^{\sigma y},$$