Spring 2007
Math 642
(Narcowich)

Preservation of Extremals under Coörd. Change

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Change of coörds. Start w/ $\underline{x}$. Fn'l is $J(\underline{x}) = \int_a^b F(t, \underline{x}, \dot{\underline{x}})\,dt$. Change of ~~variables~~ variables here means replacing $\underline{x}$ by $\underline{x} = \Phi(\underline{y})$. (Note, $\underline{y} = \Psi(\underline{x})$, w/ Ψ, Φ being ~~mutually~~ inverse functs.)

Make the replacement $J(\underline{x}) = J(\Phi(\underline{y}))$.

$$J(\Phi(\underline{y})) = \int_a^b F(t, \Phi(\underline{y}), D\Phi(\underline{y})\,\dot{\underline{y}})\,dt$$

~~Define~~ Define $\tilde{J}(\underline{y}) = J(\Phi(\underline{y}))$, so

$$\tilde{J}(\underline{y}) = \int_c^b \tilde{F}(t, \underline{y}, \dot{\underline{y}})\,dt,$$

where $\tilde{F}(t, \underline{y}, \dot{\underline{y}}) = F(t, \Phi(\underline{y}), D\Phi(\underline{y})\,\dot{\underline{y}})$

Let's calculate $\Delta\tilde{J}(\underline{y}; \underline{\nu}) = \frac{d}{d\varepsilon}\left(\tilde{J}(\underline{y} + \varepsilon\underline{\nu})\right)\Big|_{\varepsilon=0}$.

Now, $\Delta\tilde{J}(\underline{y}; \underline{\nu}) = \frac{d}{d\varepsilon}\left(J(\Psi(\underline{y} + \varepsilon\underline{\nu}))\right)\Big|_{\varepsilon=0}$

$$= \int_a^b \frac{\partial}{\partial\varepsilon}\left\{F(t, \Phi(\underline{y} + \varepsilon\underline{\nu}), D\Phi(\underline{y} + \varepsilon\underline{\nu})(\dot{\underline{y}} + \varepsilon\dot{\underline{\nu}})\right\}\Big|_{\varepsilon=0} dt$$

$$\underset{\uparrow}{=} \int_a^b \left\{\frac{\partial F}{\partial \underline{x}} D\Phi(\underline{y})\underline{\nu} + \frac{\partial F}{\partial \dot{\underline{x}}} \underbrace{\frac{\partial}{\partial\varepsilon}\left\{D\Phi(\underline{y} + \varepsilon\underline{\nu})(\dot{\underline{y}} + \varepsilon\dot{\underline{\nu}})\right\}}_{\frac{\partial}{\partial\varepsilon}\left\{\frac{\partial}{\partial t}\Phi(\underline{y} + \varepsilon\underline{\nu})\right\}}\right\}\Bigg|_{\varepsilon=0} dt$$

Lots of chain rule here!

$$= \int_a^b \left[\frac{\partial F}{\partial \underline{x}} D\Phi(\underline{y})\underline{\nu} + \frac{\partial F}{\partial \dot{\underline{x}}} \frac{d}{dt}\left\{\frac{\partial}{\partial\varepsilon}(\Phi(\underline{y} + \varepsilon\underline{\nu}))\right\}\right] dt$$

$$= \int_a^b \left\{ \frac{\partial F}{\partial \underline{x}} D\Phi(\underline{y})\underline{\nu} + \frac{\partial F}{\partial \dot{\underline{x}}} \frac{d}{dt}\left(D\Phi(\underline{y})\underline{\nu}\right) \right\} dt$$

$$= \int_a^b \left\{ \frac{\partial F}{\partial \underline{x}} D\Phi(\underline{y})\underline{\nu} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\underline{x}}}\right) D\Phi(\underline{y})\underline{\nu} \right\} dt$$

$$= \int_a^b \left\{ \frac{\partial F}{\partial \underline{x}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\underline{x}}}\right)\right\} D\Phi(\underline{y})\underline{\nu}(\underline{y})\, dt$$

If we let $\underline{\eta}(\underline{x}) = \overbrace{D\Phi(\underline{y})}^{\text{Matrix}}\ \overbrace{\underline{\nu}(\underline{y})}^{\text{Col. vect.}}$, $\begin{cases} \underline{x} = \phi(\underline{y}) \\ \underline{y} = \Psi(\underline{x}) \end{cases}$,

then we have shown that

$$\boxed{\Delta \tilde{J}(\underline{y}, \underline{\nu}) = \Delta J(\underline{x}, \underline{\eta})},$$

where $\underline{\eta}(\underline{x}) = D\Phi(\underline{y})\, \underline{\nu}(\underline{y})$ — and $\begin{cases} x = \phi(\underline{y}) \\ y = \Psi(\underline{x}) \end{cases}$.

We note that $D\phi(\underline{y}) = D\Psi(x)^{-1}$, so

$$\underline{\eta}(\underline{x}) = (D\Psi(\underline{x}))^{-1}\underline{\nu}(\underline{y}) \Rightarrow \nu(\underline{y}) = D\Psi(x)\, \underline{\eta}(x).$$

<u>Extremals</u> Because of the $\underline{\eta} \Leftrightarrow \underline{\nu}$ connection above, $\Delta\tilde{J}(\underline{y}, \underline{\nu}) = 0 \ \forall\ \underline{\nu}$ iff $\Delta J(\underline{x}, \underline{\eta}) = 0$ $\forall\ \underline{\eta}$. Hence, $\underline{x}(t)$ being an extremal of $J \Rightarrow$ $\underline{y}(t) := \Psi(\underline{x}(t))$ is an extremal of $\tilde{J}$.

<u>Point</u>: Extremals are preserved under a change of coördinates.