Coordinate Vectors and Examples

Coordinate vectors. This is a brief discussion of coordinate vectors and the notation for them that was presented in class. Here is the setup for all of the problems. We begin with a vector space $V$ that has a basis $B = \{v_1, \ldots, v_n\}$. We always keep the same order for vectors in the basis. Technically, this is called an ordered basis. If $v \in V$, then we can always express $v \in V$ in exactly one way as a linear combination of the the vectors in $B$. Specifically, for any $v \in V$ there are scalars $x_1, \ldots, x_n$ such that
\[ v = x_1v_1 + x_2v_2 + \cdots + x_nv_n. \] (1)
The $x_j$’s are the coordinates of $v$ relative to $B$. We collect them into the coordinate vector
\[ [v]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \]
We remark that the operation of finding coordinates is a linear one. That is, we have that
\[ [v + w] = [v] + [w] \quad \text{and} \quad [cv] = c[v]. \]
This is a very important property and allows us to deal use matrix methods in connection with solving problems in finite dimensional spaces.

Examples. Here are some examples. Let $V = \mathcal{P}_2$ and $B = \{1, x, x^2\}$. What is the coordinate vector $[5 + 3x - x^2]_B$? Answer:
\[ [5 + 3x - x^2]_B = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}. \]
If we ask the same question for $[5 - x^2 + 3x]_B$, the answer is the same, because to find the coordinate vector we have to order the basis elements so that they are in the same order as $B$.

Let’s turn the question around. Suppose that we are given
\[ [p]_B = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \]
then what is $p$? Answer: $p(x) = 3 \cdot 1 + 0 \cdot x + (-4) \cdot x^2 = 3 - 4x^2$.

Let’s try another space. Let $V = \text{span}\{e^t, e^{-t}\}$, which is a subspace of $C(-\infty, \infty)$. Here, we will take $B = \{e^t, e^{-t}\}$. What are coordinate vectors for $\sinh(t)$ and $\cosh(t)$? Solution: Since $\sinh(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ and $\cosh(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$, these vectors are

$$[\sinh(t)]_B = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad [\cosh(t)]_B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$  

There is an important special case. Suppose that the vector space $V$ is an inner product space and the basis $B$ is an orthogonal set of vectors – i.e., $\langle \mathbf{v}_k, \mathbf{v}_j \rangle = 0$ if $j \neq k$ and $\|\mathbf{v}_j\|^2 > 0$. From (1), we have

$$\langle \mathbf{v}, \mathbf{v}_j \rangle = \langle x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_n \mathbf{v}_n, \mathbf{v}_j \rangle = x_1 \langle \mathbf{v}_1, \mathbf{v}_j \rangle + x_2 \langle \mathbf{v}_2, \mathbf{v}_j \rangle + \cdots + x_n \langle \mathbf{v}_n, \mathbf{v}_j \rangle = x_j \|\mathbf{v}_j\|^2.$$  

This gives us the following: Relative to an orthogonal basis, the coordinate $x_j$ is given by

$$x_j = \frac{\langle \mathbf{v}, \mathbf{v}_j \rangle}{\|\mathbf{v}_j\|^2}.$$  

Matrices for linear transformations. The matrix that represents a linear transformation $L : V \to W$, where $V$ and $W$ are vector spaces with bases $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $D = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, respectively, is easy to get.

We start with the equation $\mathbf{w} = L(\mathbf{v})$. Express $\mathbf{v}$ in terms of the basis $B$ for $V$: $\mathbf{v} = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_n \mathbf{v}_n$. Next, apply $L$ to both sides of this equation and use the fact that $L$ is linear to get

$$\mathbf{w} = L(\mathbf{v}) = L(x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_n \mathbf{v}_n) = x_1 L(\mathbf{v}_1) + x_2 L(\mathbf{v}_2) + \cdots + x_n L(\mathbf{v}_n).$$  

Now, take $C$ coordinates of both sides of $\mathbf{w} = x_1 L(\mathbf{v}_1) + x_2 L(\mathbf{v}_2) + \cdots + x_n L(\mathbf{v}_n)$:

$$[\mathbf{w}]_D = [x_1 L(\mathbf{v}_1) + x_2 L(\mathbf{v}_2) + \cdots + x_n L(\mathbf{v}_n)]_D = x_1[L(\mathbf{v}_1)]_D + x_2[L(\mathbf{v}_2)]_D + \cdots + x_n[L(\mathbf{v}_n)]_D = A\mathbf{x},$$  

where the columns of $A$ are the coordinate vectors $[L(\mathbf{v}_j)]_D$, $j = 1, \ldots, n$.  

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A matrix example. Let $V = W = \mathcal{P}_2$, $B = D = \{1, x, x^2\}$, and $L(p) = ((1 - x^2)p')'$. To find the matrix $A$ that represents $L$, we first apply $L$ to each of the basis vectors in $B$.

$L(1) = 0$, $L(x) = -2x$, and $L(x^2) = 2 - 6x^2$.

Next, we find the $D$-basis coordinate vectors for each of these. Since $B = D$ here, we have

\[
[0]_D = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad [-2x]_D = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \quad [2 - 6x^2]_D = \begin{pmatrix} 2 \\ 0 \\ -6 \end{pmatrix},
\]

and so the matrix that represents $L$ is

\[
A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}.
\]

Changing coordinates. We are frequently faced with the problem of replacing a set of coordinates relative to one basis with a set for another. Let $B = \{v_1, \ldots, v_n\}$ and $B' = \{v'_1, \ldots, v'_n\}$ be bases for a vector space $V$. If $v \in V$, then it has coordinate vectors relative to each basis, $x = [v]_B$ and $x' = [v]_{B'}$. This means that

\[
v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = x'_1 v'_1 + x'_2 v'_2 + \cdots + x'_n v'_n.
\]

If we take coordinates relative to $B$ on both sides, then we arrive at the chain of equations below:

\[
x = x'_1 [v'_1]_B + x'_2 [v'_2]_B + \cdots + x'_n [v'_n]_B \Rightarrow \left( [v'_1]_B \cdots [v'_n]_B \right) x' = Cx'.
\]

Of course, we also have $x' = C^{-1}x$. We point out that these matrices have the forms below:

\[
C = C_{B \rightarrow B'} = \begin{bmatrix} [B' \text{ basis }] \\ B \end{bmatrix} \quad \text{and} \quad C^{-1} = C_{B' \rightarrow B} = \begin{bmatrix} [B \text{ basis }] \\ B' \end{bmatrix}.
\]