Methods for Finding Bases

1 Bases for the subspaces of a matrix

Row-reduction methods can be used to find bases. Let us now look at an example illustrating how to obtain bases for the *row space*, *null space*, and *column space* of a matrix A. To begin, we look at an example, the matrix A on the left below. If we row reduce A, the result is U on the right.

$$A = \begin{pmatrix} \mathbf{1} & \mathbf{1} & 2 & 0 \\ \mathbf{2} & \mathbf{4} & 2 & 4 \\ \mathbf{2} & \mathbf{1} & 5 & -2 \end{pmatrix} \Longleftrightarrow U = \begin{pmatrix} \mathbf{1} & 0 & 3 & -2 \\ 0 & \mathbf{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Let the rows of A be denoted by $\mathbf{r}_j$, $j = 1, 2, 3$, and the columns of A by $\mathbf{a}_k$, $k = 1, 2, 3, 4$. Similarly, ρ_j denotes the rows of U. (We will not need the columns of U.)

1.1 Row space

The *row* spaces of A and U are identical. This is because elementary row operations preserve the span of the rows and are themselves reversible operations. Let's see in detail how this works for A. The row operations we used to row reduce A are these.

$$\begin{array}{rccccc} \textit{step } 1: & R_2 & = & R_2 - 2R_1 & = & \mathbf{r}_2 - 2\mathbf{r}_1 \\ & R_3 & = & R_3 - 2R_1 & = & \mathbf{r}_3 - 2\mathbf{r}_1 \\ \textit{step } 2: & R_3 & = & R_3 + \frac{1}{2}R_2 & = & \mathbf{0} \\ \textit{step } 3: & R_2 & = & \frac{1}{2}R_2 & = & \frac{1}{2}\mathbf{r}_2 - \mathbf{r}_1 \\ \textit{step } 4: & R_1 & = & R_1 - R_2 & = & 2\mathbf{r}_1 - \frac{1}{2}\mathbf{r}_2 \end{array}$$

Inspecting these row operations shows that the rows of U satisfy

$$\rho_1 = 2\mathbf{r}_1 - \frac{1}{2}\mathbf{r}_2 \quad \rho_2 = \frac{1}{2}\mathbf{r}_2 - \mathbf{r}_1 \quad \rho_3 = \mathbf{0}.$$

It's not hard to run the row operations backwards to get the rows of A in terms of those of U.

$$\mathbf{r}_1 = \rho_1 + \rho_2 \quad \mathbf{r}_2 = 2\rho_1 + 4\rho_2 \quad \mathbf{r}_3 = 2\rho_1 + \rho_2.$$

Thus we see that the nonzero rows of U span the row space of A.

They are also linearly independent. To test this, we begin with the equation

$$c_1\rho_1 + c_2\rho_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

Inserting the rows in the last equation we get

$$\begin{pmatrix} c_1 & c_2 & 3c_1 - c_2 & -2c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}.$$

This gives us $c_1 = c_2 = 0$, so the rows are linearly independent. Since they also span the row space of A, they form a basis for the row space of A. This is a general fact. *The nonzero rows in the row reduced form U of a matrix A form a basis for the row space of A.*

1.2 Null space

We recall that the null space of A and U are also identical. The equations we get from finding the null space of U – *i.e.*, solving $U\mathbf{x} = \mathbf{0}$ – are

$$\begin{aligned} x_1 + 3x_3 - 2x_4 &= 0 \\ x_2 - x_3 + 2x_4 &= 0. \end{aligned}$$

The leading variables correspond to the columns containing the leading entries, which are in boldface in U in (1); these are the variables x_1 and x_2. The remaining variables, x_3 and x_4, are free (nonleading) variables. To emphasize this, we assign them new labels, $x_3 = t_1$ and $x_4 = t_2$. Solving the system obtained above, we get

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t_1 \underbrace{\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\mathbf{n}_1} + t_2 \underbrace{\begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{n}_2} = t_1\mathbf{n}_1 + t_2\mathbf{n}_2\,. \tag{2}$$

From this equation, it is easy to show that the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ form a basis for the null space. Notice that we can get these vectors by solving $U\mathbf{x} = \mathbf{0}$ first with $t_1 = 1, t_2 = 0$ and then with $t_1 = 0, t_2 = 1$.

This works in the general case as well. To find a basis for the null space, begin by identifying the leading variables $x_{\ell_1}, x_{\ell_2}, \ldots, x_{\ell_r}$, where r is the number of leading variables, and the free variables $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$. For the free variables, let $t_j = x_{f_j}$. Find the $n - r$ solutions to $U\mathbf{x} = \mathbf{0}$ corresponding to the free-variable choices $t_1 = 1, t_2 = 0, \ldots, t_{n-r} = 0$, $t_1 = 1, t_2 = 0, \ldots, t_{n-r} = 0$, $\ldots$, $t_1 = 0, t_2 = 0, \ldots, t_{n-r} = 1$. Call these vectors $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{n-r}$. *The set $\{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{n-r}\}$ is a basis for the null space of A (and, of course, U).*

1.3 Column space

We now turn to finding a basis for the column space of the matrix A in (1). The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ given in (2) span the null space of A, so they satisfy $A\mathbf{n}_1 = \mathbf{0}$ and $A\mathbf{n}_2 = \mathbf{0}$. Writing these two vector equations using the "basic matrix trick" gives us:

$$-3\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0} \quad \text{and} \quad 2\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0}\,.$$

We can use these to solve for the free columns in terms of the leading columns,

$$\mathbf{a}_3 = 3\mathbf{a}_1 - \mathbf{a}_2 \quad \text{and} \quad \mathbf{a}_4 = -2\mathbf{a}_1 + 2\mathbf{a}_2\,.$$

Thus the column space is spanned by the set $\{\mathbf{a}_1, \mathbf{a}_2\}$. ($\mathbf{a}_1$ and $\mathbf{a}_2$ are in boldface in the matrix A above.) This set is also linearly independent because the equation

$$0 = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + 0\mathbf{a}_3 + 0\mathbf{a}_4 = A\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix}$$

implies that $\begin{pmatrix} x_1 & x_2 & 0 & 0 \end{pmatrix}^T$ is in the null space of A. Matching this vector with the general form of a vector in the null space shows that the corresponding t_1 and t_2 are 0, and therefore so are x_1 and x_2. It follows that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent. Since it spans the columns as well, it is a basis for the *column space* of A. Note that these columns correspond to the *leading* variables in the problems, x_1 and x_2. This is no accident. The argument that we used can be employed to show that this is true in general: *In a matrix A, the columns of A that correspond to the leading variables in the associated homogeneous problem, $U\mathbf{x} = \mathbf{0}$, form a basis for the column space of A.*

2 Another matrix example

Let's do another example. Consider the matrix A and the matrix U, its row reduced form, shown below.

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 & 3 \\ -2 & -1 & 2 & 1 & 1 \\ -1 & 2 & 1 & 3 & 4 \\ 0 & 5 & 0 & 5 & 7 \end{pmatrix} \Longleftrightarrow U = \begin{pmatrix} 1 & 0 & -1 & -1 & -\frac{6}{5} \\ 0 & 1 & 0 & 1 & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From U, we can read off a basis for the row space,

$$\left\{\begin{pmatrix} 1 & 0 & -1 & -1 & -\frac{6}{5} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & \frac{7}{5} \end{pmatrix}\right\}.$$

Again, from U we see that the leading variables are x_1 and x_2, so the leading columns in A are $\mathbf{a}_1$ and $\mathbf{a}_2$. Thus, a basis for the column space is the set

$$\left\{\begin{pmatrix} 1 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \\ 5 \end{pmatrix}\right\}.$$

To get a basis for the null space, note that the free variables are x_3 through x_5. Let $t_1 = x_3$, etc. The system corresponding to $U\mathbf{x} = \mathbf{0}$ then has the form

$$\begin{aligned} x_1 - t_1 - t_2 - \frac{6}{5}t_3 &= 0 \\ x_2 + t_2 + \frac{7}{5}t_3 &= 0\,. \end{aligned}$$

To get $\mathbf{n}_1$, set $t_1 = 1$, $t_2 = t_3 = 0$ and solve for x_1 and x_2. This gives us $\mathbf{n}_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix}^T$. For $\mathbf{n}_2$, set $t_1 = 0$, $t_2 = 1$, $t_3 = 0$, in the system above; the result is $\mathbf{n}_2 = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \end{pmatrix}^T$. Last, set $t_1 = 0$, $t_2 = 0$, $t_3 = 1$ to get $\mathbf{n}_3 = \begin{pmatrix} \frac{6}{5} & -\frac{7}{5} & 0 & 0 & 1 \end{pmatrix}^T$. The basis for the null space is thus

$$\left\{\mathbf{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{n}_3 = \begin{pmatrix} \frac{6}{5} \\ -\frac{7}{5} \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

We want to make a few remarks on this example, concerning the dimensions of the spaces involved. The common dimension of both the row space and the column space is $\text{rank}(A) = 2$, which is also the number of leading variables. The dimension of the null space is the nullity of A. Here, $\text{nullity}(A) = 3$. Thus, in this case we have verified that

$$\text{rank}(A) + \text{nullity}(A) = 5,$$

the number of columns of A.