Final Examination

Instructions: Show all work in your bluebook. No calculators that can graph or do linear algebra are allowed.

1. (8 pts.) Determine whether or not the set \( \{1 + x, 1 - x, x + x^2, x^2 - x^3\} \) is linearly independent. Is it a basis for \( P_3 \)?

2. (5 pts.) Given that \( S \) is the set of all polynomials \( p \in P_3 \) such that \( p'(0) - p'(1) = 0 \), determine whether or not \( S \) is a subspace of \( P_3 \).

3. (7 pts.) Let \( (u, v, w) = f(x, y) = (xy, 3x - 2y, x^2y) \) and also let \( (s, t) = g(u, v, w) = (uw, v^2 + w^2) \). Use the chain rule to find the affine approximation to \( g \circ f(x, y) \) at \( x = y = 1 \).

4. Let \( L : P_2 \to P_2 \) be given by \( L[p] := (1 - t^2)p'' + (1 - 3t)p' + 8p \)

   (a) (4 pts.) Show that \( L \) is linear.

   (b) (7 pts.) Find the matrix of \( L \) relative to the basis \( E = \{1, t, t^2\} \).

   (c) (7 pts.) Find a basis for the null space (kernel) of \( L \). Is \( L \) one-to-one? onto?

5. Consider the bases for \( P_2 \) defined by

   \[ E = \{1, t, t^2\} \quad \text{and} \quad B = \{t - t^2, t + t^2, 1 + t\} \, . \]

   (a) (6 pts.) Find the change of basis matrix \( C \) that takes coördinates relative to \( B \) into ones relative to \( E \).

   (b) (6 pts.) Find the change of basis matrix \( C \) that takes coördinates relative to \( E \) into ones relative to \( B \).

   (c) (5 pts.) For the linear transformation \( L \) in problem 4, write out the form of the matrix for \( L \) relative to \( B \), given your answer to 4b.

6. (5 pts.) Find the eigenvalues and eigenvectors for the matrix

   \[ B = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \, . \]
7. **(10 pts.)** Use the Gram-Schmidt process to find an orthonormal basis for the span of the vectors below.

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} & v_2 &= \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} & v_3 &= \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}
\end{align*}
\]

![Fig. 1: The Surface S](image)

8. **(15 pts.)** The surface \( S \) in figure 1 is given parameterically by

\[
x(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (4 - r^2) \mathbf{k}, \quad 0 \leq r \leq 2, \quad 0 \leq \theta \leq \pi/4.
\]

Let \( \mathbf{G}(\mathbf{x}) = (2z - y) \mathbf{i} + (x + 3z) \mathbf{j} + (2x + 3y) \mathbf{k} \). Use Stokes’s Theorem to find the line integral

\[
\Psi = \oint_C \mathbf{G}(\mathbf{x}) \cdot d\mathbf{x},
\]

where \( C \) is the boundary of \( S \) traversed from \( A \rightarrow B \rightarrow C \rightarrow A \).

9. Let \( \Sigma \) be the surface of the sphere with center \((0, 0, 0)\) and radius 2, and let \( \mathbf{n} \) be the outward drawn normal to \( \Sigma \).

   (a) **(5 pts.)** Let \( \mathbf{F} = xy^2 \mathbf{i} + yz^2 \mathbf{j} + zx^2 \mathbf{k} \), and let \( \mathbf{n} \) be the unit normal to \( \Sigma \). Set up, but do not evaluate, an iterated double integral that gives the surface integral

   \[
   \Phi = \iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} \, dS.
   \]

   (b) **(10 pts.)** Use Gauss’s Theorem to convert \( \Phi \) into a volume integral, and then evaluate the volume integral.