Final Examination

Instructions: *Show all work in your bluebook. No calculators that can graph or do linear algebra are allowed.*

1. **(8 pts.)** Determine whether or not the set $\{1+x, 1-x, x+x^2, x^2-x^3\}$ is linearly independent. Is it a basis for $\mathcal{P}_3$?

2. **(5 pts.)** Given that S is the set of all polynomials $p \in \mathcal{P}_3$ such that $p'(0) - p'(1) = 0$, determine whether or not S is a *subspace* of $\mathcal{P}_3$.

3. **(7 pts.)** Let $(u, v, w) = f(x, y) = (xy, 3x - 2y, x^2y)$ and also let $(s, t) = g(u, v, w) = (uw, v^2 + w^2)$. Use the chain rule to find the affine approximation to $g \circ f(x, y)$ at $x = y = 1$.

4. Let $L : \mathcal{P}_2 \to \mathcal{P}_2$ be given by $L[p] := (1 - t^2)p'' + (1 - 3t)p' + 8p$

   (a) **(4 pts.)** Show that L is *linear*.

   (b) **(7 pts.)** Find the matrix of L relative to the basis $\mathcal{E} = \{1, t, t^2\}$.

   (c) **(7 pts.)** Find a basis for the null space (kernel) of L. Is L one-to-one? onto?

5. Consider the bases for $\mathcal{P}_2$ defined by

$$\mathcal{E} = \{1, t, t^2\} \quad \text{and} \quad \mathcal{B} = \{t - t^2, t + t^2, 1 + t\}\,.$$

   (a) **(6 pts.)** Find the change of basis matrix C that takes coördinates relative to $\mathcal{B}$ into ones relative to $\mathcal{E}$.

   (b) **(6 pts.)** Find the change of basis matrix C that takes coördinates relative to $\mathcal{E}$ into ones relative to $\mathcal{B}$.

   (c) **(5 pts.)** For the linear transformation L in problem 4, write out the form of the matrix for L relative to $\mathcal{B}$, given your answer to 4b.

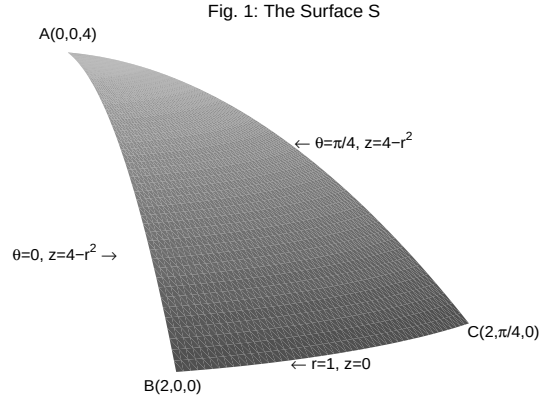
6. **(5 pts.)** Find the eigenvalues and eigenvectors for the matrix

$$B = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}.$$

7. **(10 pts.)** Use the Gram-Schmidt process to find an orthonormal basis for the span of the vectors below.

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Fig. 1: The Surface S



8. **(15 pts.)** The surface S in figure 1 is given parameterically by

$$\mathbf{x}(r,\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + (4-r^2)\mathbf{k}\,,\ 0 \le r \le 2,\ 0 \le \theta \le \pi/4.$$

Let $\mathbf{G}(\mathbf{x}) = (2z-y)\,\mathbf{i} + (x+3z)\,\mathbf{j} + (2x+3y)\mathbf{k}$. Use Stokes's Theorem to find the line integral

$$\Psi = \oint_C \mathbf{G}(\mathbf{x}) \cdot d\mathbf{x},$$

where C is the boundary of S traversed from A $\to$ B $\to$ C $\to$ A.

9. Let Σ be the surface of the sphere with center $(0,0,0)$ and radius 2, and let $\mathbf{n}$ be the outward drawn normal to Σ.

   (a) **(5 pts.)** Let $\mathbf{F} = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$, and let $\mathbf{n}$ be the unit normal to Σ. Set up, but do not evaluate, an iterated double integral that gives the surface integral

$$\Phi = \iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS\,.$$

   (b) **(10 pts.)** Use Gauss's Theorem to convert Φ into a volume integral, and then evaluate the volume integral.