Quiz 1 – Key

Instructions: Show all work in the space provided. No notes, calculators, cell phones, etc. are allowed.

1. Define the terms below.
   (a) (5 pts.) function $f$ from a set $X$ to a set $Y$ – p. 2.
   (b) (5 pts.) well-ordering principle – p. 13.

2. (15 pts.) Prove that if $|x| \leq 1$, then $|x^2 - x - 2| \leq 3|x + 1|$

   Solution. Note that $|x^2 - x - 2| = |(x - 2)(x + 1)| = |x - 2||x + 1|$. By the triangle inequality and $|x| \leq 1$, we have $|x - 2| \leq |x| + 2 \leq 3$. Hence, $|x^2 - x - 2| \leq 3|x - 1|$.

3. (10 pts.) Use the binomial theorem to show that if $a$ and $b$ are nonnegative real numbers, then $(a + b)^n \geq a^n + na^{n-1}b$.

   Solution. The binomial theorem gives us this chain:

   \[
   (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\
   = a^n + na^{n-1}b + \text{nonneg. terms} \\
   \geq a^n + na^{n-1}b.
   \]

4. (15 pts.) (Approximation Property for Suprema). Prove this:
   If $E \subseteq \mathbb{R}$ has a supremum $s$, then for every $\varepsilon > 0$ there is an $a \in E$ such that $s - \varepsilon < a \leq s$.

   Proof. Suppose not. Then for some $\varepsilon_0 > 0$ the interval $(s - \varepsilon_0, s]$ contains no points from $E$. Since $s$ is the supremum for $E$, there are no points of $E$ in $(s, \infty)$, either. It follows that all $a \in E$ are in $(-\infty, s - \varepsilon_0]$. Hence, $s - \varepsilon_0$ is an upper bound for $E$. However, $s - \varepsilon_0 < s$. This is a contradiction, since every upper bound for $E$ is greater than or equal to $s$, the supremum.