Quiz 3 – Key

Instructions: Show all work in the space provided. No notes, calculators, cell phones, etc. are allowed.

1. Define the terms below.

   (a) (5 pts.) \( \lim_{x \to a} f(x) \) does not exist – Let \( L \in \mathbb{R} \). We will first define \( \lim_{x \to a} f(x) \neq L \): For some \( \epsilon_0 > 0 \) and every \( \delta > 0 \) there is an \( x \) satisfying \( 0 < |x - a| < \delta \) for which \( |f(x) - L| \geq \epsilon_0 \). For the limit not to exist, this must hold for all \( L \).

   (b) (5 pts.) \( \lim_{x \to a^-} f(x) = L \) – p. 66

2. (15 pts.) Show that if \( |x_{n+1} - x_n| \leq 2^{-n} \) then \( x_n \) is convergent. (Hint: show that it is a Cauchy sequence.)

   Solution. First, note that this chain of inequalities holds:

   \[
   |x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n| \\
   \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
   \leq 2^{-m+1} + 2^{-m+2} + \cdots + 2^{-n} \quad \text{(assumption)} \\
   \leq 2^{-n}(1 + 2^{-1} + 2^{-2} + \cdots 2^{-(m-n-1)}) \\
   \leq 2^{-n} \frac{1 - 2^{-(m-n)}}{1/2} = 2(2^{-n} - 2^{-m}) \quad \text{(geometric series)}
   \]

   For \( \epsilon/4 > 0 \), choose \( N \in \mathbb{N} \) such that \( 2^{-n} < \epsilon/4 \) when \( n \geq N \). It follows form the last inequality if \( n, m \geq N \) we have that \( |x_m - x_n| \leq 2(2^{-n} - 2^{-m}) < 2(2^{-n} + 2^{-m}) < \epsilon \). Hence, \( \{x_n\} \) is a Cauchy sequence and is therefore convergent.

3. (10 pts.) Show: \( f \vee g(x) := \max\{f(x), g(x)\} = \frac{(f+g(x)+|f-g(x)|)}{2} \).

   Solution. Assume \( f(x) \geq g(x) \), so \( f \vee g(x) = f(x) \). Then, we also have \( |(f-g)(x)| = f(x) - g(x) \), and so

   \[
   \frac{(f + g)(x) + |(f - g)(x)|}{2} = \frac{f(x) + g(x) + f(x) - g(x)}{2} = f(x) = f \vee g(x)
   \]
4. (15 pts.) (Sequential Characterization of Limits) Prove this: Let $a \in I \subseteq \mathbb{R}$, where $I$ is open, and let $f : I \setminus \{a\} \rightarrow \mathbb{R}$. If $f(x_n) \rightarrow L$ for every sequence $x_n \in I \setminus \{a\}$ such that $x_n \rightarrow a$, then $L = \lim_{x \to a} f(x)$ exists.

**Proof.** Suppose not. Then, for some $\varepsilon_0 > 0$ and every $\delta$ there is an $x$ such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon_0$. Take $\delta = 1, 1/2, \ldots, 1/n, \ldots$ for each choice of $\delta$, we have $x_n$ such that $0 < |x_n - a| < 1/n$ and $|f(x_n) - L| \geq \varepsilon_0$. By the squeeze theorem for sequences, $x_n \rightarrow a$. Thus, from our assumption, $f(x_n) \rightarrow L$. Equivalently, $\lim_{n \to \infty} |f(x_n) - L| = 0$. However, by the comparison theorem, $\lim_{n \to \infty} |f(x_n) - L| \geq \varepsilon_0 > 0$. This is a contradiction, so $\lim_{x \to a} f(x) = L$. 