Final Examination

Instructions: Show all work in your bluebook. You may use a calculator for numerical computations. You may not use a graphing calculator or a calculator that can do symbolics.

1. (15 pts.) Let $A$ be the circulant matrix below, and let $a = (-2, 1, 0, 1)$. Verify that if $y = Ax$, then $y = a ∗ x$. Use the DFT and the (circular) convolution theorem to find the eigenvalues of $A$. (Hint: $w = -i$ in this case.)

$$A = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

2. (20 pts.) For $j ∈ \mathbb{Z}$, let $V_j$ be the subspace of $f ∈ L^2$ such that the support of $\hat{f}$ being in $[-2^j π, 2^j π]$, and let $φ(x) = \text{sinc}(x)$; note that $φ ∈ V_0$. List all of the properties of a multiresolution analysis (MRA). Pick any three; show that the $V_j$’s and $φ$ satisfy them.

3. (15 pts.) For any MRA, the reconstruction formula is given by $a^{j+1}_k = \sum_{ℓ ∈ \mathbb{Z}} p_k 2^{-j} a^{j}_k + \sum_{ℓ ∈ \mathbb{Z}} (-1)^k P_{-k+2ℓ} b^{j+1}_ℓ$. Put this formula in terms of the discrete filters shown in Figure 1. State what $\tilde{H}, \tilde{L}$, and $2↑$ are. For the Daubechies wavelet, are these filters IIR or FIR?

4. (20 pts.) Let $φ(x)$ be the Haar scaling function. Use the Haar MRA to decompose to level 0 the function $f ∈ V_2$, where

$$f(x) = 2φ(4x + 1) - φ(4x) + 3φ(4x - 1) + 5φ(4x - 2) - φ(4x - 3).$$

5. (15 pts.) The scaling relation for an MRA is $φ(x) = \sum_{k=-∞}^{∞} p_k φ(2x - k)$. Show that this becomes, in the Fourier transform picture, $\hat{φ}(ξ) = P(e^{-iξ/2}) \hat{φ}(ξ/2)$, where $P(z) = \frac{1}{2} \sum_{k=-∞}^{∞} p_z z^k$. Briefly describe how $\hat{φ}(ξ)$ is constructed from the function $P(ξ)$. What conditions should $P(z)$ satisfy for this construction to yield a scaling function?
6. (15 pts.) For the Daubechies $N = 2$ MRA, the function $P(z)$ is a polynomial,

$$P(z) = (1 + z)^2 \left( \frac{1 + \sqrt{3}}{8} + \frac{1 - \sqrt{3}}{8}z \right).$$

Use this and the formula $\hat{\psi}(\xi) = -e^{-i\xi/2} P(-e^{-i\xi/2}) \hat{\phi}(\xi/2)$ to show that the Daubechies wavelet has two vanishing moments. Briefly discuss the significance of this for singularity detection.

Properties of the Fourier Transform

1. $\hat{f}(\xi) = \mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$.

2. $f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi$.

3. $\mathcal{F}[x^n f(x)](\xi) = i^n \hat{f}^{(n)}(\xi)$.

4. $\mathcal{F}[f^{(n)}(x)](\xi) = (i\xi)^n \hat{f}(\xi)$.

5. $\mathcal{F}[f(x-a)](\xi) = e^{-i\xi a} \hat{f}(\xi)$.

6. $\mathcal{F}[f(bx)](\xi) = \frac{1}{b} \hat{f} \left( \frac{\xi}{b} \right)$.

7. $\mathcal{F}[f * g] = \sqrt{2\pi} \hat{f}(\xi) \hat{g}(\xi)$

8. $\text{sinc}(x) := \frac{\sin(\pi x)}{\pi x} = \mathcal{F}^{-1}[\chi_{\pi}]$, where \( \chi_{\pi}(\xi) = \left\{ \begin{array}{ll} 1/\sqrt{2\pi}, & -\pi \leq \xi \leq \pi \\ 0, & |\xi| > \pi \end{array} \right. $.