Math 414-500 (Spring 2002) Name __________________________ 1

Test I

Instructions: Show all work in your bluebook. You may use a calculator for numerical computations. You may not use a graphing calculator or a calculator that can do symbolics.

1. (15 pts.) For \( n > 0 \), let \( f_n(t) = \begin{cases} 1, & 0 \leq t \leq 1/n, \\ 0, & \text{otherwise}. \end{cases} \) Show that \( f_n \to 0 \) in \( L^2[0,1] \). Show that \( f_n \) does not converge to zero uniformly on \([0,1]\).

2. (20 pts.) Use least squares to fit a straight line to the data below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>7.0</td>
<td>4.2</td>
<td>-2.1</td>
<td>-5.0</td>
</tr>
</tbody>
</table>

3. Let \( h(x) := \pi/2 - x \), \( 0 \leq x \leq \pi \).

   (a) (10 pts.) Sketch two periods each of the functions to which the Fourier cosine series (FCS) and Fourier sine series (FSS) converge pointwise. Are either of these series uniformly convergent on \([0,\pi]\)? How about on \([\pi/4,3\pi/4]\)? Why?

   (b) (15 pts.) Find the FCS for \( h \). Use it to evaluate the sum \( \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \).

   (c) (5 pts.) On \([0,\pi]\), \( h \) has a symmetry property. State it. Use it to explain why all of the even coefficients in the FCS for \( h \) vanish.

4. (20 pts.) Let \( \sigma \) be real, and not an integer. Find the complex form of the Fourier series for the \( 2\pi \)-periodic function \( F \), where \( F(x) = e^{-i\sigma x} \) on \(-\pi < x < \pi \). Use this and Parseval’s theorem to show that

\[
\csc^2(\sigma \pi) = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(\sigma + k)^2}.
\]

5. (15 pts.) Do one of the following.

   (a) Sketch a proof for this theorem. Suppose \( f \) is a continuous and \( 2\pi \)-periodic function. Then for each point \( x \), where the derivative of \( f \) is defined, the Fourier series of \( f \) at \( x \) converges to \( f(x) \).

   (b) Recall that the FS for \( g(x) = \begin{cases} \pi - x, & 0 \leq x \leq \pi, \\ -\pi - x, & -\pi \leq x < 0 \end{cases} \) exhibits the Gibbs phenomenon near \( x = 0 \). Briefly describe what this is for \( g \), and then show that it is universal.
Integrals

1. \( \int udv = uv - \int vdu \)
2. \( \int \frac{dt}{t} = \ln |t| + C \)
3. \( \int e^{at} dt = \frac{1}{a} e^{at} + C \)
4. \( \int t^n e^{at} dt = \frac{1}{a} t^n e^{at} - \frac{n}{a} \int t^{n-1} e^{at} dt \)
5. \( \int e^{at} \cos(bt) dt = \frac{e^{at}}{a^2 + b^2} (a \cos(bt) + b \sin(bt)) + C \)
6. \( \int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2 + b^2} (a \sin(bt) - b \cos(bt)) + C \)
7. \( \int t \sin(t) dt = \sin(t) - t \cos(t) + C \)
8. \( \int t \cos(t) dt = \cos(t) + t \sin(t) + C \)
9. \( \int \sin(at) dt = -\frac{1}{a} \cos(at) + C \)
10. \( \int \cos(at) dt = \frac{1}{a} \sin(at) + C \)
11. \( \int \tan(at) dt = \frac{1}{a} \ln |\sec(at)| + C \)
12. \( \int \cot(at) dt = \frac{1}{a} \ln |\sin(at)| + C \)
13. \( \int \sec(at) dt = \frac{1}{a} \ln |\sec(at) + \tan(at)| + C \)
14. \( \int \csc(at) dt = \frac{1}{a} \ln |\csc(at) - \cot(at)| + C \)
15. \( \int \frac{dt}{t^2 + a^2} = \frac{1}{a} \arctan(t/a) + C \)
16. \( \int \frac{dt}{t^2 - a^2} = \frac{1}{2a} \ln \left| \frac{t - a}{t + a} \right| + C \)