Test I

Instructions: Show all work in your bluebook. You may use a calculator for numerical computations. You may not use a graphing calculator or a calculator that can do symbols.

1. Let \( g(x) := \frac{\pi}{2} - x, \) \( 0 \leq x \leq \pi \).
   
   (a) (10 pts.) Find the FCS for \( g \), and sketch three periods of the function to which it converges pointwise.
   
   (b) (5 pts.) Sketch three periods of the function to which the Fourier sine series converges pointwise. (Do \emph{not} compute the coefficients in the series.)
   
   (c) (10 pts.) Define the term \emph{uniform convergence}. Is either series uniformly convergent? If so, which? Why? Will either series exhibit the Gibbs’ phenomenon? Briefly explain.

2. (10 pts.) Let \( r \) be real. Show that the complex form of the Fourier series for the \( 2\pi \)-periodic function \( F(x) = e^{rx} \) on \( -\pi < x < \pi \), is

   \[
   F(x) = \frac{e^{2\pi r} - e^{-2\pi r}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+\pi} e^{inx}.
   \]

   Use this and Parseval’s theorem to sum the series \( \sum_{n=-\infty}^{\infty} \frac{1}{(n+\pi)^2} \).

3. Consider the function \( f(t) = e^{-|t|} \). Do the following.
   
   (a) (10 pts.) Show that \( \hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\lambda^2} \).
   
   (b) (5 pts.) Find the integral \( \int_{-\infty}^{\infty} \frac{du}{(1+u^2)^2} \).
   
   (c) (5 pts.) Find the transform \( \mathcal{F}[tf(t-1)] \).

4. Let \( h(t) = \begin{cases} Ae^{-\alpha t} & t \geq 0, \\ 0 & t < 0 \end{cases} \) be the impulse response (IR) for the Butterworth filter \( H[f] = h * f \).

   (a) (10 pts.) Find \( H[f] \), where \( f(t) = \begin{cases} 1 & 0 \leq t \leq 2, \\ 0 & t < 0 \text{ or } t > 2 \end{cases} \)

   (b) (5 pts.) Define the term \emph{causal filter}. Is \( H \) causal? Explain.

5. (15 pts.) Let \( S_n \) be the space of \( n \) periodic sequences. If \( y \in S_n \) and if \( z \in S_n \) is defined by \( z_j = y_{j+1} \), show that \( \hat{z}_k = w^k \hat{y}_k \), where \( w = e^{2\pi i/n} \).

6. (15 pts.) Do one of the following.

   (a) State and prove the Riemann-Lebesgue Lemma in the case where \( f(x) \) is continuously differentiable on the finite interval \([a, b]\).

   (b) State and prove the Sampling Theorem.

   (c) Sketch the proof for the pointwise convergence of the Fourier series of a function \( f \) that is \( 2\pi \)-periodic, piecewise continuous, and has a piecewise continuous derivative.
Fourier Transform Properties

1. \( \hat{f}(\xi) = F[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi} \, dx. \)
2. \( f(x) = F^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi} \, d\xi. \)
3. \( F[x^n f(x)](\xi) = i^n \hat{f}^{(n)}(\xi). \)
4. \( F[f^{(n)}](\xi) = (i\xi)^n \hat{f}(\xi). \)
5. \( F[f(x-a)](\xi) = e^{-i\xi a} \hat{f}(\xi). \)
6. \( F[f(bx)](\xi) = \frac{1}{b} \hat{f}\left(\frac{\xi}{b}\right). \)
7. \( F[f \ast g] = \sqrt{2\pi} \hat{f}(\xi) \hat{g}(\xi) \)

Integrals

1. \( \int udv = uv - \int vdu \)
2. \( \int \frac{dt}{t} = \ln |t| + C \)
3. \( \int e^{at} \, dt = \frac{1}{a} e^{at} + C \)
4. \( \int t^n e^{at} \, dt = \frac{1}{a} t^n e^{at} - \frac{n}{a} \int t^{n-1} e^{at} \, dt \)
5. \( \int e^{at} \cos(bt) \, dt = \frac{e^{at}}{a^2 + b^2} (a \cos(bt) + b \sin(bt)) + C \)
6. \( \int e^{at} \sin(bt) \, dt = \frac{e^{at}}{a^2 + b^2} (a \sin(bt) - b \cos(bt)) + C \)
7. \( \int t \sin(t) \, dt = \sin(t) - t \cos(t) + C \)
8. \( \int t \cos(t) \, dt = \cos(t) + t \sin(t) + C \)
9. \( \int \tan(at) \, dt = \frac{1}{a} \ln |\sec(at)| + C \)
10. \( \int \cot(at) \, dt = \frac{1}{a} \ln |\sin(at)| + C \)
11. \( \int \sec(at) \, dt = \frac{1}{a} \ln |\sec(at) + \tan(at)| + C \)
12. \( \int \csc(at) \, dt = \frac{1}{a} \ln |\csc(at) - \cot(at)| + C \)
13. \( \int \frac{dt}{t^2 + a^2} = \frac{1}{a} \arctan(t/a) + C \)