Notes on Row Reduction

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1 Notation and Terms

The row echelon form of a matrix contains a great deal of information, both about the matrix itself and about systems of equations that may be associated with it. To talk about row reduction, we need to define several terms and introduce some notation.

1. *Notation for elementary row operations.* Row reduction is made easier by having notation for various row operations. This is the notation that we will use here. (The equals sign ""="" used in the row operations below means "replacement," which is common usage in computer languages.)

    I. $R \leftrightarrow R'$ means interchange rows R and R'.

    II. $R = cR$ means multiply the current row R by $c \neq 0$ and make the result the new row R.

    III. $R = R + cR'$ means multiply R' by c, then add cR' to R, and make the result the new row R.

2. *Row equivalence.* A matrix A is *row equivalent* to a matrix B if A can be transformed into B using elementary row operations. When this happens, we write $A \Leftrightarrow B$.

3. *Leading entry.* The *leading entry*[1] in a row is the first non-zero entry in a row. The leading entries in each row of M are in boldface type.

$$M = \begin{pmatrix} 0 & \mathbf{1} & 3 & 2 \\ \mathbf{2} & 4 & 0 & -1 \\ 0 & 0 & \mathbf{6} & 5 \end{pmatrix}$$

4. *Pivot and pivotal row.* See the row reduction algorithm below.

5. *Row echelon form of a matrix.* A matrix is in *row echelon form* if these hold:

   (a) In any nonzero row, the leading (first nonzero) entry is 1.

   (b) If row k has a leading entry, then either the leading entry in row k+1 is to the right of the one for row k or row k+1 is a zero row.

   (c) All rows with only zeros for entries are at the bottom.

6. *Reduced row echelon form of a matrix.* A matrix is in *reduced row echelon form* if these hold:

   (a) The matrix is in row echelon form.

   (b) The leading entry in a row is the only nonzero entry in its column. That is, any column containing a leading entry has zeros in all other entries.

The matrix M above is *not* in row echelon form. Here are two examples illustrating the two row-echelon forms. The first matrix below is in row echelon form, but not *reduced* row echelon form, while the second is in reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

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[1]Leon does not have a name for this entry. However, most authors do use the term *leading entry.*

2 The Row Reduction Algorithm

Although we will not prove it here, every matrix is row equivalent to a matrix that is in reduced row echelon form. We call the process of finding the (reduced) echelon form of a matrix *row reduction.* Below is an algorithm for row-reducing a matrix; it uses two sets of steps, *forward* ones (Gaussian elimination) and *backward* ones (Jordan reduction). The *forward steps* obtain a row echelon form of the matrix; the *backward steps* then give the reduced row echelon for of the matrix.

Forward Steps

1. Starting on the left, find the first column that contains a leading entry. If there are several leading entries in this column, choose a convenient one—for example, an entry that is 1. The leading entry is now called the *pivot* and this row is called the *pivotal row.*

2. Interchange rows until the pivotal row is on top.

3. Use the top row and elementary modification to zero-out the other entries in the column that you are working with.

4. Repeat the first three steps with the submatrix comprising all columns to the right of the one you were working with, and all rows below the top. Stop when either there are no more rows left or the next submatrix consists of zeros. Divide each row by its pivot. When this step is complete, all rows with only zeros for entries are at the bottom of the matrix. All leading entries are 1. The matrix is then in *row echelon form.*

Backward Steps

1. Find the leading entry in the last non-zero row. Use elementary modification to zero-out all of the entries above this leading entry.

2. Repeat step one for the leading entry in the second to the last non-zero row. Once this is done, do the same thing for the third to the last row, fourth to last, and so on. When there are no more rows left, the matrix is in reduced row echelon form.

An example. We will now row reduce a matrix using the algorithm given above. Consider the matrix A shown below.

$$\text{(Starting matrix) } A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & -1 & 2 \end{pmatrix}$$

This is the matrix we will row reduce. We will begin by carrying out the forward steps in the algorithm. Leading entries are in boldface type.

$$R_2 = R_2 - 2R_1, \; R_3 = R_3 - R_1 : \quad A \Leftrightarrow \begin{pmatrix} \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -\mathbf{2} & 1 \end{pmatrix}$$

$$R_2 \leftrightarrow R_3 : \quad A \Leftrightarrow \begin{pmatrix} \mathbf{1} & 1 & 1 \\ 0 & -\mathbf{2} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_2 = -\frac{1}{2}R_2 : \quad A \Leftrightarrow \begin{pmatrix} \mathbf{1} & 1 & 1 \\ 0 & \mathbf{1} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

This ends the set of forward steps. The matrix is now in *row echelon* form, but not in *reduced* row echelon form.

To put it in reduced row echelon form, we will need to perform the backward steps.

$$R_1 = R_1 - R_2 : \quad A \Leftrightarrow \begin{pmatrix} \mathbf{1} & 0 & \frac{3}{2} \\ 0 & \mathbf{1} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

This completes the algorithm. The last matrix on the right above is the reduced row echelon form of A.

3 Solving Systems of Equations

We now want to look at how to solve a linear system of equations using row reduction to implement the method of elimination. We begin with the system below.

$$\begin{aligned} x_1 - 3x_2 - x_3 - 3x_4 &= 0 \\ -x_1 + 3x_2 + 2x_3 + 4x_4 &= 1 \\ 2x_1 - 6x_2 + 4x_3 &= 6 \end{aligned}$$

The coefficient matrix A and "right-hand side" b for this system are

$$A = \begin{pmatrix} 1 & -3 & -1 & -3 \\ -1 & 3 & 2 & 4 \\ 2 & -6 & 4 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix},$$

and the augmented matrix $[A|b]$ is given by

$$[A|b] = \left(\begin{array}{cccc|c} 1 & -3 & -1 & -3 & 0 \\ -1 & 3 & 2 & 4 & 1 \\ 2 & -6 & 4 & 0 & 6 \end{array} \right).$$

To solve the original system, we row reduce the augmented matrix:

$$R_2 = R_2 + R_1, \ R_3 = R_3 - 2R_1 : \quad [A|b] \Leftrightarrow \left(\begin{array}{cccc|c} 1 & -3 & -1 & -3 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 & 6 \end{array} \right)$$

$$R_3 = R_2 - 6R_2 : \quad [A|b] \Leftrightarrow \left(\begin{array}{cccc|c} \mathbf{1} & -3 & -1 & -2 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{Backward step:} \quad R_1 = R_1 + R_2, \ : \quad [A|b] \Leftrightarrow \left(\begin{array}{cccc|c} \mathbf{1} & -3 & 0 & -2 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The final matrix on the right above is actually the reduced echelon form of $[A|b]$. The underlined matrix elements are the leading entries. Identifying them tells us that the *leading variables* are x_1 and x_3, the *nonleading variables* are x_2 and x_4, and the rank of the augmented matrix is 2. The equivalent system reduces to two equations, $x_1 - 3x_2 - 2x_4 = 1$ and $x_3 + x_4 = 1$. Assigning parameters to the *nonleading* variables and solving for the *leading* variables, we end up with the solution in *parametric form*:

$$\begin{cases} x_1 &= 1 + 3t_1 + 2t_2 \\ x_2 &= t_1 \\ x_3 &= 1 - t_2 \\ x_4 &= t_2 \end{cases}.$$