

# Applied Analysis Qualifying Exam

May 21, 2008

**Instructions:** Do any 7 of the 9 problems in this exam. Show all of your work clearly. Please indicate which 2 of the 9 problems you are skipping.

**Policy on misprints:** The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem in such a way that it becomes trivial.

1. Consider the function  $I(x) := \int_0^1 t^x(2-t)^x dt$ . Use Laplace's method and Watson's lemma to find the asymptotic expansion  $I(x)$  to order  $\mathcal{O}(x^{-2})$ , for  $x \rightarrow \infty$ .
2. Let  $\mathcal{S}$  be Schwartz space,  $\mathcal{S}'$  be the space of tempered distributions, and let  $T(x) = (x)_+ - 3(x-2)_+ + 2(x-3)_+$ , where  $(x)_+ = \begin{cases} x & 0 \leq x \\ 0 & x < 0 \end{cases}$ .
  - (a) Define  $\mathcal{S}$  and give the semi-norm topology for it. In addition, define  $\mathcal{S}'$ .
  - (b) Find the distributional second derivative,  $T''$ , and the Fourier transform of  $T''$ . Use these results to find the Fourier transform of  $T$ .
3. Consider a functional  $J[u]$ , where  $u \in V$ , and  $V$  is a Banach space.
  - (a) Define the Frechét derivative and the Gâteaux derivative for  $J[u]$ . Illustrate the difference between them with a simple two variable example.
  - (b) Consider the constrained functional,

$$J[u] = \int_0^1 pu'^2 dx + \sigma u(1)^2, \quad H[u] = \int_0^1 u^2 dx = 1,$$

where  $u \in C^{(1)}[0, 1]$ ,  $u(0) = 0$ , and  $\sigma > 0$ . Calculate the variational derivative of the problem, using Lagrange multipliers. Find the Sturm-Liouville eigenvalue problem associated with it.

- (c) How does the second eigenvalue of this problem compare with the second eigenvalue of the corresponding Dirichlet problem, i.e.,  $u(0) = u(1) = 0$ ? Explain your answer.
4. Let  $\mathcal{H}$  be a Hilbert space with inner product and norm given by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ . You may assume that  $\mathcal{H}$  is a real Hilbert space.
- (a) State and prove the Riesz Representation Theorem.
- (b) Suppose that  $\mathcal{H} \subset C[0, 1]$  and that, for  $f \in \mathcal{H}$ ,  $\|f\|_{C[0,1]} \leq \|f\|$ . Show that for every  $\xi \in [0, 1]$  there is a function  $K_\xi(\cdot) \in \mathcal{H}$  for which  $f(\xi) = \langle f, K_\xi \rangle$ .
- (c) Consider  $N$  distinct points  $0 \leq \xi_1 < \xi_2 < \dots < \xi_N \leq 1$  and let  $\mathcal{U} := \text{span}\{K_{\xi_j}\}_{j=1}^N$ , which is a finite dimensional subspace of  $\mathcal{H}$ . Show that for any  $f \in \mathcal{H}$ , the orthogonal projection of  $f$  onto  $\mathcal{U}$ ,  $\tilde{f} = \text{Proj}_{\mathcal{U}} f$ , satisfies  $\tilde{f}(\xi_j) = f(\xi_j)$ ,  $j = 1, \dots, N$ .
5. The degree  $n$  Chebyshev polynomial can be defined via the Rodrigues' formula,

$$T_n(x) = (-2)^n \frac{n!}{(2n)!} (1-x^2)^{1/2} \frac{d^n}{dx^n} ([1-x^2]^{n-1/2}).$$

- (a) Using the Rodrigues' formula, show that, in the inner product  $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)(1-x^2)^{-1/2} dx$ ,  $T_n$  is orthogonal to all polynomials of degree  $n-1$  or less.
- (b) Show that the generating function for the Chebyshev polynomials is

$$\Phi(x, w) := \sum_{n=0}^{\infty} T_n(x)w^n = \frac{1-xw}{1-2xw+w^2}.$$

6. Do *one* of the following.
- (a) State the Weierstrass approximation theorem and sketch a proof.
- (b) Let  $\mathcal{H}$  be a complex, separable Hilbert space with inner product and norm given by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ . If  $L$  is a self-adjoint operator defined on a domain  $D \subseteq \mathcal{H}$ , show that  $L$  has no residual spectrum and that its spectrum is real.

7. Consider the Schrödinger operator  $Hu = -u'' + q(x)u$ , with a compactly supported, continuous potential  $q(x) \geq 0$ . Show that the left and right transmission coefficients are equal; that is,  $T_L(k) = T_R(k)$ .
8. Let  $L$  to be the self-adjoint operator  $Lu = -u''$ , where  $D(L) = \{u \in L^2(\mathbb{R}) : u'' \in L^2(\mathbb{R})\}$ .

- (a) Find the Green's function for  $L$ .
- (b) Employ Stone's formula (i.e., the spectral theorem for self-adjoint operators) to obtain the Fourier transform,

$$F(\mu) = \int_{-\infty}^{\infty} f(x)e^{i\mu x} dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\mu)e^{-i\mu x} d\mu.$$

9. Let  $\mathcal{H}$  be a complex (separable) Hilbert space, with  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  being the inner product and norm.
- (a) Let  $\lambda \in \mathbb{C}$  be fixed. If  $K : \mathcal{H} \rightarrow \mathcal{H}$  is a compact linear operator, show that the range of the operator  $L = I - \lambda K$  is closed.
- (b) Briefly explain why the operator  $Ku(x) = \int_0^\pi \sin(x-t)u(t)dt$  is compact on  $L^2[0, \pi]$ .
- (c) Determine the values of  $\lambda$  for which  $u = f + \lambda Ku$  has a solution for all  $f \in \mathcal{H}$ , given that  $K$  is the operator in 9b.